Lecture 5: Singular Value Decomposition

- singular value decomposition
- matrix norms
- linear systems
- LS, pseudo-inverse, orthogonal projections
- low-rank matrix approximation
- singular value inequalities
- computing the SVD via the power method
Main Results

• any matrix $A \in \mathbb{R}^{m \times n}$ admits a singular value decomposition

$$A = U\Sigma V^T,$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{m \times m}$ are orthogonal, and $\Sigma \in \mathbb{R}^{m \times n}$ has $[\Sigma]_{ij} = 0$ for all $i \neq j$ and $[\Sigma]_{ii} = \sigma_i$ for all $i$, with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min\{m,n\}}$.

• matrix 2-norm: $\|A\|_2 = \sigma_1$

• let $r$ be the number of nonzero $\sigma_i$'s, partition $U = [U_1 \ U_2]$, $V = [V_1 \ V_2]$, and let $\tilde{\Sigma} = \text{Diag}(\sigma_1, \ldots, \sigma_r)$

  – pseudo-inverse: $A^\dagger = V_1 \tilde{\Sigma}^{-1} U_1^T$

  – LS solution: $x_{LS} = A^\dagger y + \eta$ for any $\eta \in \mathcal{R}(V_2)$

  – orthogonal projection: $P_A = U_1 U_1^T$
Main Results

- low-rank matrix approximation: given $A \in \mathbb{R}^{m \times n}$ and $k \in \{1, \ldots, \min\{m, n\}\}$, the problem

$$\min_{B \in \mathbb{R}^{m \times n}, \text{rank}(B) \leq k} \|A - B\|_F^2$$

has a solution given by $B^* = \sum_{i=1}^{k} \sigma_i u_i v_i^T$
Singular Value Decomposition

**Theorem 5.1.** Given any $A \in \mathbb{R}^{m \times n}$, there exists a 3-tuple $(U, \Sigma, V) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times n} \times \mathbb{R}^{n \times n}$ such that

$$A = U \Sigma V^T,$$

$U$ and $V$ are orthogonal, and $\Sigma$ takes the form

$$[\Sigma]_{ij} = \begin{cases} 
\sigma_i, & i = j \\
0, & i \neq j
\end{cases}, \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0, \ p = \min\{m, n\}.$$

- the above decomposition is called the **singular value decomposition (SVD)**
- $\sigma_i$ is called the $i$th **singular value**
- $u_i$ and $v_i$ are called the $i$th **left and right singular vectors**, resp.
- the following notations may be used to denote singular values of a given $A$

$$\sigma_{\text{max}}(A) = \sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_p(A) = \sigma_{\text{min}}(A)$$
Different Ways of Writing out SVD

• partitioned form: let \( r \) be the number of nonzero singular values, and note \( \sigma_1 \geq \ldots \sigma_r > 0, \sigma_{r+1} = \ldots = \sigma_p = 0 \). Then,

\[
A = [U_1 \quad U_2] \begin{bmatrix}
\tilde{\Sigma} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
V_1^T \\
V_2^T
\end{bmatrix},
\]

where
- \( \tilde{\Sigma} = \text{Diag}(\sigma_1, \ldots, \sigma_r) \),
- \( U_1 = [u_1, \ldots, u_r] \in \mathbb{R}^{m \times r}, U_2 = [u_{r+1}, \ldots, u_m] \in \mathbb{R}^{m \times (m-r)} \),
- \( V_1 = [v_1, \ldots, v_r] \in \mathbb{R}^{n \times r}, V_2 = [v_{r+1}, \ldots, v_n] \in \mathbb{R}^{n \times (n-r)} \).

• thin SVD: \( A = U_1 \tilde{\Sigma} V_1^T \)

• outer-product form: \( A = \sum_{i=1}^{p} \sigma_i u_i v_i^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T \)
SVD and Eigendecomposition

From the SVD $A = U\Sigma V^T$, we see that

$$AA^T = UD_1U^T, \quad D_1 = \Sigma \Sigma^T = \text{Diag}(\sigma_1^2, \ldots, \sigma_p^2, 0, \ldots, 0)$$

$$A^TA = VD_2V^T, \quad D_2 = \Sigma^T\Sigma = \text{Diag}(\sigma_1^2, \ldots, \sigma_p^2, 0, \ldots, 0)$$

Observations:

- (*) and (**) are the eigendecompositions of $AA^T$ and $A^TA$, resp.
- the left singular matrix $U$ of $A$ is the eigenvector matrix of $AA^T$
- the right singular matrix $V$ of $A$ is the eigenvector matrix of $A^TA$
- the squares of nonzero singular values of $A$, $\sigma_1^2, \ldots, \sigma_r^2$, are the nonzero eigenvalues of both $AA^T$ and $A^TA$. 
Insights of the Proof of SVD

• the proof of SVD is constructive

• to see the insights, consider the special case of square nonsingular \( A \)

\[ A A^T \text{ is PD, and denote its eigendecomposition by} \]

\[ A A^T = U \Lambda U^T, \quad \text{with} \quad \lambda_1 \geq \ldots \geq \lambda_n > 0. \]

• let \( \Sigma = \text{Diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}) \), \( V = A^T U \Sigma^{-1} \)

• it can be verified that \( U \Sigma V^T = A \), \( V^T V = I \)

• see the accompanying note for the proof of SVD in the general case
**SVD and Subspace**

**Property 5.1.** The following properties hold:

(a) $\mathcal{R}(A) = \mathcal{R}(U_1)$, $\mathcal{R}(A)^\perp = \mathcal{R}(U_2)$;
(b) $\mathcal{R}(A^T) = \mathcal{R}(V_1)$, $\mathcal{R}(A^T)^\perp = \mathcal{N}(A) = \mathcal{R}(V_2)$;
(c) $\text{rank}(A) = r$ (the number of nonzero singular values).

Note:

- in practice, SVD can be used as a numerical tool for computing bases of $\mathcal{R}(A)$, $\mathcal{R}(A)^\perp$, $\mathcal{R}(A^T)$, $\mathcal{N}(A)$
- we have previously learnt the following properties
  - $\text{rank}(A^T) = \text{rank}(A)$
  - $\text{dim} \mathcal{N}(A) = n - \text{rank}(A)$

By SVD, the above properties are easily seen to be true.
Matrix Norms

- the definition of a norm of a matrix is the same as that of a vector:
  \[ f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \]
  is a norm if (i) \( f(A) \geq 0 \) for all \( A \); (ii) \( f(A) = 0 \) if and only if \( A = 0 \); (iii) \( f(A + B) \leq f(A) + f(B) \) for any \( A, B \); (iv) \( f(\alpha A) = |\alpha| f(A) \)
  for any \( \alpha, A \)

- naturally, the Frobenius norm \( \|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = [\text{tr}(A^T A)]^{1/2} \) is a norm

- there are many other matrix norms

- induced norm or operator norm: the function
  \[ f(A) = \max_{\|x\|_\beta \leq 1} \|Ax\|_\alpha \]

  where \( \| \cdot \|_\alpha, \| \cdot \|_\beta \) denote any vector norms, can be shown be to a norm
Matrix Norms

- matrix norms induced by the vector $p$-norm ($p \geq 1$):

$$\|A\|_p = \max_{\|x\|_p \leq 1} \|Ax\|_p$$

- it is known that

- $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$
- $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$

- how about $p = 2$?
Matrix 2-Norm

- matrix 2-norm or spectral norm:

\[ \|A\|_2 = \sigma_{\text{max}}(A). \]

- proof:
  - for any \( x \) with \( \|x\|_2 \leq 1 \),
    \[
    \|Ax\|_2^2 = \|U\Sigma V^T x\|_2^2 = \|\Sigma V^T x\|_2^2 \\
    \leq \sigma_1^2 \|V^T x\|_2^2 = \sigma_1^2 \|x\|_2^2 \leq \sigma_1^2
    \]
  - \( \|Ax\|_2 = \sigma_1 \) if we choose \( x = v_1 \)

- implication to linear systems: let \( y = Ax \) be a linear system. Under the input energy constraint \( \|x\|_2 \leq 1 \), the system output energy \( \|y\|_2^2 \) is maximized when \( x \) is chosen as the 1st right singular vector

- corollary: \( \min_{\|x\|_2=1} \|Ax\|_2 = \sigma_{\text{min}}(A) \) if \( m \geq n \)
Matrix 2-Norm

Properties for the matrix 2-norm:

- $\|AB\|_2 \leq \|A\|_2 \|B\|_2$
  - in fact, $\|AB\|_p \leq \|A\|_p \|B\|_p$ for any $p \geq 1$

- $\|Ax\|_2 \leq \|A\|_2 \|x\|_2$
  - a special case of the 1st property

- $\|QAW\|_2 = \|A\|_2$ for any orthogonal $Q, W$
  - we also have $\|QAW\|_F = \|A\|_F$ for any orthogonal $Q, W$

- $\|A\|_2 \leq \|A\|_F \leq \sqrt{p} \|A\|_2$ (here $p = \min\{m, n\}$)
  - proof: $\|A\|_F = \|\Sigma\|_F = \sqrt{\sum_{i=1}^{p} \sigma_i^2}$, and $\sigma_1^2 \leq \sum_{i=1}^{p} \sigma_i^2 \leq p\sigma_1^2$
Schatten $p$-Norm

• the function

$$f(A) = \left( \min\{m,n\} \sum_{i=1}^{\min\{m,n\}} \sigma_i(A)^p \right)^{1/p}, \quad p \geq 1,$$

is known to be a norm and is called the Schatten $p$-norm (how to prove it?).

• nuclear norm:

$$\|A\|_* = \sum_{i=1}^{\min\{m,n\}} \sigma_i(A)$$

– a special case of the Schatten $p$-norm

– a way to prove that the nuclear norm is a norm:
  * show that $f(A) = \max_{\|B\|_2 \leq 1} \text{tr}(B^T A)$ is a norm
  * show that $f(A) = \sum_{i=1}^{\min\{m,n\}} \sigma_i$ is a norm

– finds applications in rank approximation, e.g., for compressive sensing and matrix completion [Recht-Fazel-Parrilo’10]
Schatten $p$-Norm

- $\text{rank}(A)$ is nonconvex in $A$ and is arguably hard to do optimization with it.

- **Idea:** the rank function can be expressed as

  $$\text{rank}(A) = \sum_{i=1}^{\min\{m,n\}} 1\{\sigma_i(A) \neq 0\},$$

  and why not approximate it by

  $$f(A) = \sum_{i=1}^{\min\{m,n\}} \varphi(\sigma_i(A))$$

  for some friendly function $\varphi$?

- **nuclear norm**

  $$\|A\|_* = \sum_{i=1}^{\min\{m,n\}} \sigma_i(A)$$

  - uses $\varphi(z) = z$
  
  - is convex in $A$
Linear Systems: Sensitivity Analysis

• **Scenario:**
  
  – let \( A \in \mathbb{R}^{n \times n} \) be nonsingular, and \( y \in \mathbb{R}^n \). Let \( x \) be the solution to
  
  \[ y = Ax. \]
  
  – consider a perturbed version of the above system: \( \hat{A} = A + \Delta A, \hat{y} = y + \Delta y \), where \( \Delta A \) and \( \Delta y \) are errors. Let \( \hat{x} \) be a solution to the perturbed system
  
  \[ \hat{y} = \hat{A}\hat{x}. \]

• **Problem:** analyze how the solution error \( \| \hat{x} - x \|_2 \) scales with \( \Delta A \) and \( \Delta y \)

• **remark:** \( \Delta A \) and \( \Delta y \) may be floating point errors, measurement errors, etc.
Linear Systems: Sensitivity Analysis

- the condition number of a given matrix $A$ is defined as

$$
\kappa(A) = \frac{\sigma_{\text{max}}(A)}{\sigma_{\text{min}}(A)},
$$

- $\kappa(A) \geq 1$, and $\kappa(A) = 1$ if $A$ is orthogonal

- $A$ is said to be ill-conditioned if $\kappa(A)$ is very large; that refers to cases where $A$ is close to singular
Linear Systems: Sensitivity Analysis

**Theorem 5.2.** Let $\varepsilon > 0$ be a constant such that

$$
\frac{\| \Delta A \|_2}{\| A \|_2} \leq \varepsilon, \quad \frac{\| \Delta y \|_2}{\| y \|_2} \leq \varepsilon.
$$

If $\varepsilon$ is sufficiently small such that $\varepsilon \kappa(A) < 1$, then

$$
\frac{\| \hat{x} - x \|_2}{\| x \|_2} \leq \frac{2\varepsilon \kappa(A)}{1 - \varepsilon \kappa(A)}.
$$

- **Implications:**
  - for small errors and in the worst-case sense, the relative error $\| \hat{x} - x \|_2 / \| x \|_2$ tends to increase with the condition number
  - in particular, for $\varepsilon \kappa(A) \leq \frac{1}{2}$, the error bound can be simplified to

$$
\frac{\| \hat{x} - x \|_2}{\| x \|_2} \leq 4\varepsilon \kappa(A)
$$
Linear Systems: Interpretation under SVD

• consider the linear system

\[ y = Ax \]

where \( A \in \mathbb{R}^{m \times n} \) is the system matrix; \( x \in \mathbb{R}^n \) is the system input; \( y \in \mathbb{R}^m \) is the system output

• by SVD we can write

\[ y = U\tilde{y}, \quad \tilde{y} = \Sigma \tilde{x}, \quad \tilde{x} = V^T x \]

• Implication: all linear systems work by performing three processes in cascade, namely,
  
  – rotate/reflect the system input \( x \) to form an intermediate system input \( \tilde{x} \)
  
  – form an intermediate system output \( \tilde{y} \) by element-wise rescaling \( \tilde{x} \) w.r.t. \( \sigma_i \)'s and by either removing some entires of \( \tilde{x} \) or adding some zeros
  
  – rotate/reflect \( \tilde{y} \) to form the system output \( y \)
Linear Systems: Interpretation under SVD

(a) linear system

(b) equivalent system
Linear Systems: Solution via SVD

• Problem: given general $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$, determine
  – whether $y = Ax$ has a solution (more precisely, whether there exists an $x$ such that $y = Ax$);
  – what is the solution

• by SVD it can be shown that

$$y = Ax \iff y = U_1 \tilde{\Sigma} V_1^T x$$

$$\iff U_1^T y = \tilde{\Sigma} V_1^T x, \quad U_2^T y = 0$$

$$\iff V_1^T x = \tilde{\Sigma}^{-1} U_1^T y, \quad U_2^T y = 0$$

$$\iff x = V_1 \tilde{\Sigma}^{-1} U_1^T y + \eta, \text{ for any } \eta \in \mathcal{R}(V_2) = \mathcal{N}(A),$$
$$U_2^T y = 0$$
Linear Systems: Solution via SVD

- let us consider specific cases of the linear system solution characterization

\[ y = Ax \iff x = V_1 \hat{\Sigma}^{-1} U_1^T y + \eta, \text{ for any } \eta \in \mathcal{R}(V_2) = \mathcal{N}(A), \]
\[ U_2^T y = 0 \]

- Case (a): full-column rank \( A \), i.e., \( r = n \leq m \)
  - there is no \( V_2 \), and \( U_2^T y = 0 \) is equivalent to \( y \in \mathcal{R}(U_1) = \mathcal{R}(A) \)
  - Result: the linear system has a solution if and only if \( y \in \mathcal{R}(A) \), and the solution, if exists, is uniquely given by \( x = V \hat{\Sigma}^{-1} U_1^T y \)

- Case (b): full-row rank \( A \), i.e., \( r = m \leq n \)
  - there is no \( U_2 \)
  - Result: the linear system always has a solution, and the solution is given by \( x = V_1 \hat{\Sigma}^{-1} U^T y + \eta \) for any \( \eta \in \mathcal{N}(A) \)
Least Squares via SVD

• consider the LS problem

\[
\min_{x \in \mathbb{R}^n} \|y - Ax\|_2^2
\]

for general \(A \in \mathbb{R}^{m \times n}\)

• we have, for any \(x \in \mathbb{R}^n\),

\[
\|y - Ax\|_2^2 = \|y - U\Sigma V^T x\|_2^2 = \|U^T y - \Sigma \tilde{x}\|_2^2
\]

\[
= \sum_{i=1}^r |\tilde{y}_i - \sigma_i \tilde{x}_i|^2 + \sum_{i=r+1}^p |\tilde{y}_i|^2
\]

\[
\geq \sum_{i=r+1}^p |\tilde{y}_i|^2
\]

• the equality above is attained if \(\tilde{x}\) satisfies \(\tilde{y}_i = \sigma_i \tilde{x}_i\) for \(i = 1, \ldots, r\), and it can be shown that such a \(\tilde{x}\) corresponds to (try)

\[
x = V_1 \tilde{\Sigma}^{-1} U_1^T y + V_2 \tilde{x}_2, \quad \text{for any } \tilde{x}_2 \in \mathbb{R}^{n-r}
\]

which is the desired LS solution
**Pseudo-Inverse**

The pseudo-inverse of a matrix $A$ is defined as

$$A^\dagger = V_1\tilde{\Sigma}^{-1}U_1^T.$$

From the above def. we can show that

- $x_{LS} = A^\dagger y + \eta$ for any $\eta \in \mathcal{R}(V_2)$; the same applies to linear sys. $y = Ax$
- $A^\dagger$ satisfies the Moore-Penrose conditions: (i) $AA^\dagger A = A$; (ii) $A^\dagger AA^\dagger = A^\dagger$; (iii) $AA^\dagger$ is symmetric; (iv) $A^\dagger A$ is symmetric
- when $A$ has full column rank
  - the pseudo-inverse also equals $A^\dagger = (A^T A)^{-1}A^T$
  - $A^\dagger A = I$
- when $A$ has full row rank
  - the pseudo-inverse also equals $A^\dagger = A^T( AA^T)^{-1}$
  - $AA^\dagger = I$
Orthogonal Projections

• with SVD, the orthogonal projections of $y$ onto $\mathcal{R}(A)$ and $\mathcal{R}(A)\perp$ are, resp.,

$$\Pi_{\mathcal{R}(A)}(y) = Ax_{LS} = AA^\dagger y = U_1 U_1^T y$$

$$\Pi_{\mathcal{R}(A)\perp}(y) = y - Ax_{LS} = (I - AA^\dagger)y = U_2 U_2^T y$$

• the orthogonal projector and orthogonal complement projector of $A$ are resp. defined as

$$P_A = U_1 U_1^T, \quad P_A^\perp = U_2 U_2^T$$

• properties (easy to show):
  – $P_A$ is idempotent, i.e., $P_A P_A = P_A$
  – $P_A$ is symmetric
  – the eigenvalues of $P_A$ are either 0 or 1
  – $\mathcal{R}(P_A) = \mathcal{R}(A)$
  – the same properties above apply to $P_A^\perp$, and $I = P_A + P_A^\perp$
Minimum 2-Norm Solution to Underdetermined Linear Systems

- consider solving the linear system \( \mathbf{y} = \mathbf{A}\mathbf{x} \) when \( \mathbf{A} \) is fat
- this is an underdetermined problem: we have more unknowns \( n \) than the number of equations \( m \)
- assume that \( \mathbf{A} \) has full row rank. By now we know that any
  \[
  \mathbf{x} = \mathbf{A}^\dagger \mathbf{y} + \eta, \quad \eta \in \mathcal{R}(\mathbf{V}_2)
  \]
  is a solution to \( \mathbf{y} = \mathbf{A}\mathbf{x} \), but we may want to grab one solution only
- **Idea:** discard \( \eta \) and take \( \mathbf{x} = \mathbf{A}^\dagger \mathbf{y} \) as our solution
- **Question:** does discarding \( \eta \) make sense?
- **Answer:** it makes sense under the minimum 2-norm problem formulation
  \[
  \min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_2^2 \quad \text{s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}
  \]
  It can be shown that the solution is uniquely given by \( \mathbf{x} = \mathbf{A}^\dagger \mathbf{y} \) (try the proof)
Low-Rank Matrix Approximation

**Aim:** given a matrix \( A \in \mathbb{R}^{m \times n} \) and an integer \( k \) with \( 1 \leq k < \text{rank}(A) \), find a matrix \( B \in \mathbb{R}^{m \times n} \) such that \( \text{rank}(B) \leq k \) and \( B \) best approximates \( A \)

- it is somehow unclear about what a best approximation means, and we will specify one later
- closely related to the matrix factorization problem considered in Lecture 2
- applications: PCA, dimensionality reduction,...—the same kind of applications in matrix factorization
- truncated SVD: denote

  \[ A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T. \]

  Perform the aforementioned approximation by choosing \( B = A_k \)
Toy Application Example: Image Compression

- let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix whose $(i, j)$th entry $a_{ij}$ stores the $(i, j)$th pixel of an image.

- memory size for storing $\mathbf{A}$: $mn$

- truncated SVD: store $\{\mathbf{u}_i, \sigma_i \mathbf{v}_i\}_{i=1}^k$ instead of the full $\mathbf{A}$, and recover the image by $\mathbf{B} = \mathbf{A}_k$

- memory size for truncated SVD: $(m + n)k$
  - much less than $mn$ if $k \ll \min\{m, n\}$
Toy Application Example: Image Compression

(a) original image, size= 102 × 1347

(b) truncated SVD, k= 5

(c) truncated SVD, k= 10

(d) truncated SVD, k= 20
Low-Rank Matrix Approximation

• truncated SVD provides the best approximation in the LS sense:

**Theorem 5.3** (Eckart-Young-Mirsky). Consider the following problem

\[
\min_{B \in \mathbb{R}^{m \times n}, \text{rank}(B) \leq k} \| A - B \|_F^2
\]

where \( A \in \mathbb{R}^{m \times n} \) and \( k \in \{1, \ldots, p\} \) are given. The truncated SVD \( A_k \) is an optimal solution to the above problem.

• also note the matrix 2-norm version of the Eckart-Young-Mirsky theorem:

**Theorem 5.4.** Consider the following problem

\[
\min_{B \in \mathbb{R}^{m \times n}, \text{rank}(B) \leq k} \| A - B \|_2
\]

where \( A \in \mathbb{R}^{m \times n} \) and \( k \in \{1, \ldots, p\} \) are given. The truncated SVD \( A_k \) is an optimal solution to the above problem.
Low-Rank Matrix Approximation

- recall the matrix factorization problem in Lecture 2:

\[
\min_{A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times n}} \| Y - AB \|_F^2
\]

where \( k \leq \min\{m, n\} \); \( A \) denotes a basis matrix; \( B \) is the coefficient matrix

- the matrix factorization problem may be reformulated as (verify)

\[
\min_{Z \in \mathbb{R}^{m \times n}, \text{rank}(Z) \leq k} \| Y - Z \|_F^2,
\]

and the truncated SVD \( Y_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \), where \( Y = U\Sigma V^T \) denotes the SVD of \( Y \), is an optimal solution by Theorem 5.4

- thus, an optimal solution to the matrix factorization problem is

\[
A = [ u_1, \ldots, u_k ], \quad B = [ \sigma_1 v_1, \ldots, \sigma_k v_k ]^T
\]
Toy Demo: Dimensionality Reduction of a Face Image Dataset

A face image dataset. Image size $= 112 \times 92$, number of face images $= 400$. Each $x_i$ is the vectorization of one face image, leading to $m = 112 \times 92 = 10304$, $n = 400$. 
**Toy Demo: Dimensionality Reduction of a Face Image Dataset**

Mean face

1st principal left singular vector

2nd principal left singular vector

3rd principal left singular vector

400th left singular vector

![Energy Concentration Plot](chart.png)

Energy Concentration

Singular Value Inequalities

Similar to variational characterization of eigenvalues of real symmetric matrices, we can derive various variational characterization results for singular values, e.g.,

- **Courant-Fischer characterization:**
  \[
  \sigma_k(A) = \min_{\dim S_{n-k+1} \subseteq \mathbb{R}^n} \max_{x \in S_{n-k+1}, \|x\|_2 = 1} \|Ax\|_2
  \]

- **Weyl’s inequality:** for any \( A, B \in \mathbb{R}^{m \times n} \),
  \[
  \sigma_{k+l-1}(A + B) \leq \sigma_k(A) + \sigma_l(B), \quad k, l \in \{1, \ldots, p\}, \; k + l - 1 \leq p.
  \]

  Also, note the corollaries
  - \( \sigma_k(A + B) \leq \sigma_k(A) + \sigma_1(B), \; k = 1, \ldots, p \)
  - \( |\sigma_k(A + B) - \sigma_k(A)| \leq \sigma_1(B), \; k = 1, \ldots, p \)

- and many more...
Proof of the Eckart-Young-Mirsky Thm. by Weyl’s Inequality

An application of singular value inequalities is that of proving Theorem 5.4:

• for any $B$ with $\text{rank}(B) \leq k$, we have
  
  $\sigma_l(B) = 0$ for $l > k$
  
  – (Weyl) $\sigma_{i+k}(A) \leq \sigma_i(A - B) + \sigma_{k+1}(B) = \sigma_i(A - B)$ for $i = 1, \ldots, p - k$
  
  – and consequently

  $$\|A - B\|_F^2 = \sum_{i=1}^{p} \sigma_i(A - B)^2 \geq \sum_{i=1}^{p-k} \sigma_i(A - B)^2 \geq \sum_{i=k+1}^{p} \sigma_i(A)^2$$

• the equality above is attained if we choose $B = A_k$.
Computing the SVD via the Power Method

The power method can be used to compute the thin SVD, and the idea is as follows.

- Assume $m \geq n$ and $\sigma_1 > \sigma_2 > \ldots > \sigma_n > 0$

- Apply the power method to $A^TA$ to obtain $v_1$

- Obtain $u_1 = Av_1/\|Av_1\|_2, \sigma_1 = \|Av_1\|_2$ (why is this true?)

- Do deflation $A := A - \sigma_1 u_1 v_1^T$, and repeat the above steps until all singular components are found
References