This technical report provides the proof of Proposition 1 in [1] which claims that the following optimization problem is NP-hard.

$$\min_{a \in \mathbb{R}^N} a^T 1$$

$$\text{s.t.} \quad \frac{\sqrt{2|s|_{\text{max}}}}{\eta \sqrt{PA}} \rho + \frac{2|s|_{\text{max}}}{\eta} \epsilon^T a \leq g^T a$$

$$\frac{2|s|_{\text{max}}}{|s|_{\text{min}} + |s|_{\text{max}}} \|g \odot a\|_\infty \leq g^T a$$

$$a \in \{0, 1\}^N,$$

where $g \in \mathbb{R}_+^N$, $\epsilon \in \mathbb{R}_+^N$ and $\eta > \mathbb{R}_+$. For convenience, let us use the following notations:

$$s = \frac{\sqrt{2|s|_{\text{max}}}}{\eta \sqrt{PA}} \rho$$

$$t = \frac{2|s|_{\text{max}}}{|s|_{\text{min}} + |s|_{\text{max}}} - 1$$

$$p = g - \frac{2|s|_{\text{max}}}{\eta} \epsilon$$

$$I = \{i \mid a_i = 1\}.$$
Then, problem (1) can be written as

\[
\begin{align*}
\min_{\mathcal{I}} & \quad |\mathcal{I}| \\
\text{s.t.} & \quad s \leq \sum_{i \in \mathcal{I}} p_i \\
& \quad t g_i \leq \sum_{i \in \mathcal{I}, \; i \neq \bar{i}} g_i, \quad \bar{i} = \arg \max_{i \in \mathcal{I}} g_i
\end{align*}
\]

(2a) \hspace{1cm} (2b) \hspace{1cm} (2c)

\[
\mathcal{I} \subset \{0, \ldots, N\}.
\]

(2d)

Our strategy is to show that the decision version of (2) is NP-complete by reducing the knapsack problem to it. The decision version of (2) is as follows: Given a positive \(k\), determine if there exists an index set \(\mathcal{I} \subset \{1, \ldots, N\} \) such that

\[
\begin{align*}
|\mathcal{I}| \leq k \\
s & \leq \sum_{i \in \mathcal{I}} p_i \\
t g_i & \leq \sum_{i \in \mathcal{I}, \; i \neq \bar{i}} g_i, \quad \bar{i} = \arg \max_{i \in \mathcal{I}} g_i
\end{align*}
\]

(3)

The knapsack problem is as follows: Given a number \(C \in \mathbb{Z}_+\), determine if there exists an index set \(\mathcal{I}' \subset \{1, \ldots, N'\} \) such that

\[
\begin{align*}
\sum_{i \in \mathcal{I}'} c_i & \geq C \\
\sum_{i \in \mathcal{I}'} w_i & \leq W
\end{align*}
\]

(4)

where \(c \in \mathbb{Z}^{N'}_+\) and \(w \in \mathbb{Z}^{N'}_+\).

Given an instance \(J' = (c, w, C, W)\) of the knapsack problem, construct an instance \(J = (p, g, s, t, k)\) of problem (3) by

\[
\begin{align*}
g_i &= 2c_i, & p_i &= -w_i, & \text{for all } i = 1, \ldots, N' \\
g_{N' + 1} &= 1, & p_{N' + 1} &= 1 + W + \sum_{i=1}^{N'} w_i \\
g_{N' + 2} &= 2 \max_{i=1, \ldots, N'} c_i, & p_{N' + 2} &= 1 + W + 2 \sum_{i=1}^{N'} w_i \\
g_{N' + 3} &= 2C + 2 \max_{i=1, \ldots, N'} c_i, & p_{N' + 3} &= 2(1 + W + 2 \sum_{i=1}^{N'} w_i) \\
s &= 4 + 3W + 7 \sum_{i=1}^{N'} w_i \\
t &= 1 \\
N &= N' + 3, & k &= N' + 3.
\end{align*}
\]

Obviously this construction can be computed in polynomial time. We proceed to show that \(J'\) is a yes instance if and only if \(J\) is a yes instance, or equivalently the following two set of conditions are equivalent:
• Condition 1: there is an index set \( I \subset \{1, \ldots, N \} \) such that
\[
\begin{align*}
|I| & \leq k \quad (5a) \\
p_i & \leq \sum_{i \in I} p_i \quad (5b) \\
\bar{t} g_i & \leq \sum_{i \in I, \bar{i} \neq i} g_i, \quad \bar{i} = \arg \max_{i \in I} g_i. \quad (5c)
\end{align*}
\]

• Condition 2: there is an index set \( I' \subset \{1, \ldots, N' \} \) such that
\[
\begin{align*}
\sum_{i \in I'} c_i & \geq C \quad (6a) \\
\sum_{i \in I'} w_i & \leq W. \quad (6b)
\end{align*}
\]

We first show that condition 2 implies condition 1. Let \( I' \) be an index set that satisfies condition 2. Let us verify that \( I = I' \cup \{N' + 1, N' + 2, N' + 3\} \) satisfies condition 1. Clearly the (5a) is satisfied.

For (5b), consider the following inequality
\[
\sum_{i \in I} p_i = \sum_{i \in I'} p_i + \sum_{i = N' + 1}^{N' + 3} p_i \geq 4 + 4W + 7 \sum_{i = 1}^{N'} w_i - \sum_{i \in I'} w_i \\
\geq 4 + 3W + 7 \sum_{i = 1}^{N'} w_i = s,
\]
where the inequality is due to (6b).

For (5c), note that \( \bar{i} = N' + 3 \) by construction. Then, we have
\[
\sum_{i \in I, \bar{i} \neq N' + 3} g_i = \sum_{i \in I'} g_i + \sum_{i = N' + 1}^{N' + 3} g_i \\
= 1 + 2 \max_{i = 1, \ldots, N'} c_i + 2 \sum_{i \in I'} c_i \\
\geq 1 + 2 \max_{i = 1, \ldots, N'} c_i + 2C \\
\geq \bar{t} g_{N' + 3},
\]
where the first inequality is due to (6a).
We then show that condition 1 implies condition 2. Let \( \mathcal{I} \) be an index set that satisfies condition 1. Let us show that \( \{N' + 1, N' + 2, N' + 3\} \) belongs to \( \mathcal{I} \). Suppose not, then

\[
\sum_{i \in \mathcal{I}} p_i \leq p_{N' + 2} + p_{N' + 3} + \sum_{i \in \mathcal{I} \setminus \{N' + 1, N' + 2, N' + 3\}} p_i
\]

\[
= 3 \left(1 + W + 2 \sum_{i = 1}^{N'} w_i\right) - \sum_{i \in \mathcal{I} \setminus \{N' + 1, N' + 2, N' + 3\}} w_i < s,
\]

where the first inequality is due to \( p_{N' + 3} \geq p_{N' + 2} \geq p_{N' + 1} \). This result contradicts (5b). Hence, it follows have that \( \{N' + 1, N' + 2, N' + 3\} \) belongs to \( \mathcal{I}' \).

We then verify that \( \mathcal{I}' = \mathcal{I} \setminus \{N' + 1, N' + 2, N' + 3\} \) satisfies condition 2. Note that we have \( \tilde{i} = N' + 3 \) by construction. For (6a), consider

\[
\sum_{i \in \mathcal{I} \setminus \{\tilde{i}\}} g_i \geq tg_{\tilde{i}}
\]

\[
\iff g_{N' + 1} + g_{N' + 2} + \sum_{i \in \mathcal{I}'} g_i \geq g_{N' + 3}
\]

\[
\iff 2 \sum_{i \in \mathcal{I}'} c_i \geq 2C - 1
\]

\[
\iff \sum_{i \in \mathcal{I}'} c_i \geq C.
\]

where the last step is due to the fact that \( C \) and all \( c_i \) are integers. For (6b), consider

\[
\sum_{i \in \mathcal{I}} p_i \geq s
\]

\[
\iff p_{N' + 1} + p_{N' + 2} + p_{N' + 3} + \sum_{i \in \mathcal{I}'} p_i \geq s
\]

\[
\iff -\sum_{i \in \mathcal{I}'} w_i \geq -W.
\]

REFERENCES