Foundamental Course on Probability, Random Variable and Random Processes

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- Probability Theory
- Random Variables
- Random Processes
  - Stationary RP & Ergodic RP
  - Gaussian RP
  - Filtering of RP
Probability Theory - 3 stages

In order to develop a useful theory of probability, it is important to separate 3 stages in the consideration of any real probability.

1. The association of an events with a probability by (i) experiments and (ii) reasoning.

   e.g. $P(1) = \frac{1}{6}$

2. Development of the relationship of the probability of an event with the probabilities of some other events.

   e.g. $P(1) P(1)$

3. The application of the results of stage 1 & stage 2 to the real world.

   e.g.
A single performance is called a **trial** & at it we observe a single **outcome** $S_i$.

The **event** $A$ is said to have occurred in this trial if $S_i \in A$.

**Probability space** $S = \text{universal set} = \{1,2,\ldots,6\}$

Set contains all possible experimental outcomes

$\emptyset = \text{empty set} = \text{set contains impossible outcomes}$
Stage 1: The association of an event with a probability

For example, we are to determine the probability of event $A$ which is the outcome being one in a trial of throwing a dice.

- **Probability determined by experiment**

  The experiment of throwing a dice is repeated $n$ times. Suppose $n_a$ times of the $n$ trials result in event $A$.

  \[
  \text{Probability of } A = P(A) \equiv \frac{n_a}{n} \quad \text{provided } n \text{ is sufficiently large.}
  \]

  **Comments:**
  
  (i) not exact!
  
  \[
  \lim_{n \to \infty} \frac{n_a}{n} \equiv P(A) \]
  may be exact but cannot be found in practice.

- **Probability determined by reasoning**

  We may assume that throwing a dice has six possible outcomes and so there are 6 possible events. If all events have the same probability, then $P(A) = 1/6$.

  **Comment:** not exact as the assumptions may be wrong.
Stage 2: Development of the relationship between probabilities of different events

We assign to each event $A$ a number $P(A)$ which we call the probability of $A$. This number satisfies the 3 axioms:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $AB = \emptyset \rightarrow P(A + B) = P(A) + P(B)$ (i.e. mutually exclusive)

**Thm.**

1. $P(\overline{A}) = 1 - P(A)$
2. $P(\emptyset) = 0$
3. $AB \neq 0 \rightarrow P(A + B) = P(A) + P(B) - P(AB)$
   $\therefore P(AB) \leq P(A) + P(B)$
4. $B \subset A \rightarrow P(A) = P(B) + P(A\overline{B}) \geq P(B)$
Fill in the missing words in

**Definitions:**

1) Event $S$ (universal set) occurs at ____ trial.

2) Event $\emptyset$ (empty set) ____ occurs.

3) Event $A+B$ occurs when event $A$ ____ event $B$ occur.

4) Event $AB$ occurs when event $A$ ____ event $B$ occur.

**Properties:**

1) $A \cdot B = 0$ $\rightarrow$ event $A$ & event $B$ ____ occur in the same trial.

2) Event $A$ occurs $\rightarrow$ Event $\overline{A}$ ____ occurs.
Two Theorems in Probability Theory

**Prove:** \( P(\overline{A}) \leq 1 - P(A) \leq 1 \)

**Proof:**

\( A \cdot \overline{A} = \phi \) (set theory)

\( A + \overline{A} = S \) (set theory)

\[ \therefore \ P(A + \overline{A}) = P(A) + P(\overline{A}) = 1 \]  (axiom 3)

\[ \rightarrow P(\overline{A}) = 1 - P(A) \quad Q.E.D. \]

**Prove:** \( P(\phi) = 0 \)

**Proof:**

\( S = S + \overline{S} = S + \phi \) (set theory)

\( S \cdot \overline{S} = \phi \) (set theory)

\[ \therefore \ P(S + \phi) = P(S) + P(\phi) = 1 \]

\[ \rightarrow P(\phi) = 1 - P(S) = 0 \quad Q.E.D. \]
Three definitions in Probability Theory

Given event $A$ & event $B$, we have

$P(AB)$ The probability of the occurrence of events $A$ & $B$.

$P(A+B)$ The probability of the occurrence of event $A$ or $B$.

$P(A/B)$ The probability of the occurrence of event $A$ given $B$.

In general, we have

$P(ABCD....) \ &

P(A+B+C+D+....)$
**Def.** The conditional event for $A$ given $B$, $A/B$, is the event $A$ under the stipulation that $B$ has occurred.

**Def.** The conditional probability of $A$ given $B$ is $P(A/B) \equiv P(AB)/P(B)$.

**Properties**

1) $AB = 0 \rightarrow P(A/B) = 0$

2) $A \subset B \rightarrow A \cdot B = A$
   $$\rightarrow P(A/B) = \frac{P(A)}{P(B)} \geq P(A)$$

3) $B \subset A \rightarrow AB = B$
   $$\rightarrow P(A/B) = \frac{P(B)}{P(B)} = 1$$

4) $A = \{1,2,3\}$
   $B = \{1,3,5\}$
   $$\rightarrow P(A/B) = \frac{P(AB)}{P(B)} = \frac{1}{3} / \frac{1}{2} = \frac{2}{3}$$

**Examples**
**Probability Theory**

**Def.** Two events $A$ & $B$ are independent if $P(AB) \equiv P(A)P(B)$.

Two events are independent when knowledge of the occurrence of one event gives no additional information concerning the likelihood of the occurrence of a second event.

$$P(\frac{A}{B}) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

**e.g.**

**trial 1**

- $S_1 = \{1,2,3,4,5,6\}$
- $A_1 = \{1\}$
- $P(A_1) = \frac{1}{6}$

**trial 2**

- $S_2 = \{1,2,3,4,5,6\}$
- $A_2 = \{1\}$
- $P(A_2) = \frac{1}{6}$

$$P(A_1A_2) = P(A_1)P(A_2) \text{ if } A_1 \text{ & } A_2 \text{ are independent.}$$

The space of $A_1A_2$ is

$$S = S_1 \times S_2 = \{(1,1), (1,2), (1,3), \ldots, (6,6)\}.$$
Mutually Exclusive Events and Independent Events

**e.g. 1**  
\[ A = \{1\} \quad B = \{2\} \]

\[
P(A/B) = \frac{P(AB)}{P(B)} = \frac{P(\phi)}{P(B)} = 0
\]

A & B are mutually exclusive, i.e. \( AB = 0 \)

**e.g. 2**  
\[ A = \{1\} \quad B = \{2\} \]

\[
P(A/B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)
\]

A & B are independent, i.e. \( P(AB) = P(A)P(B) \)
**Independent Events**

**Def.** Events $A_1$, $A_2$ & $A_3$ are independent

iff $P(A_1A_2A_3) = P(A_1) \ P(A_2) \ P(A_3)$

& $P(A_1A_2) = P(A_1) \ P(A_2)$

& $P(A_1A_3) = P(A_1) \ P(A_3)$

& $P(A_2A_3) = P(A_2) \ P(A_3)$.

**Example:**

Given:

$P(A_1) = 1/2$

$P(A_2) = 1/4$

$P(A_3) = 1/4$

$P(A_1A_2) = 1/8$

$P(A_1A_3) = 1/8$

$P(A_2A_3) = 1/8$

$P(A_1A_2A_3) = 1/32$

Are $A_1$, $A_2$ & $A_3$ independent?

**Def.** In general, $n$ events $A_1$, $A_2$, ..., $A_n$ are independent iff

$P(A_1A_2 \ldots A_n) = P(A_1) \ P(A_2) \ldots \ P(A_n)$

: ... :

$P(A_iA_jA_k) = P(A_i) \ P(A_j) \ P(A_k)$

$P(A_iA_j) = P(A_i) \ P(A_j)$

for all combinations of $i, j, k, ...$

where $1 \leq i \leq j \leq k \leq \ldots \leq n$. 
**Thm.** For events $A_1, A_2, \ldots, A_i$ (which may or may not be independent), the probability of the simultaneous occurrence of the $i$ events is

$$P(A_1A_2\ldots A_i) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1A_2) \cdots P(A_i/A_1A_2\ldots A_{i-1}).$$

**Example:** Let $A_1, A_2, A_3$ & $A_4$ represent the consecutive events of drawing an aces.

**Find:** $P(A_1A_2A_3A_4)$

**Solution:**

$$P(A_1A_2A_3A_4) = P(A_1)P(A_2/A_1)P(A_3/A_1A_2)P(A_4/A_1A_2A_3)$$

$$= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49}$$

$$= \frac{1}{270725}$$
**PROBABILITY THEORY**

**set theory**
- \( AB \)
- \( A + B \)
- \( S \)
- \( \phi \)

**Definitions:**
- **trial**
- **outcome**
- **event**

**3 axioms**
1. \( P(A) \geq 0 \)
2. \( P(S) = 1 \)
3. \( AB = 0 \rightarrow P(A + B) = P(A) + P(B) \)

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**single trial**

**Definition:** Conditional Probability

\[
P(A / B) = \frac{P(AB)}{P(B)}
\]

**Theorem of Total Probability**

\[
P(B) = P(B / A_1)P(A_1) + \ldots + P(B / A_n) \cdot P(A_n)
\]

**multiple trials**

**Definition:** Events \( A \) & \( B \) are independent

\[
P(AB) = P(A) \cdot P(B)
\]

\[
P(A_1A_2 \ldots A_n) = \frac{P(A_1) \cdot P(A_2 / A_1)}{P(A_2 / A_1)}
\]

space \( S_1 \)

space \( S_2 \)

new space \( S = S_1 \times S_2 \)