ERG 2012B
Advanced Engineering Mathematics II

Part II: Linear Algebra

Lecture #13
Linear Systems and Determinant
Linear Dependence

Theorem 4: Linear Dependence and Independence

$p$ vectors (with $n$ components each) are linearly independent if the matrix of these row vectors has rank $p$.

They are linearly dependent if the rank is less than $p$.

Theorem 5: $p$ vectors with $n$ components each and $n < p$ are always linearly dependent.

Proof: Since each of these $p$ vectors has $n$ components, the corresponding matrix $A$ is $p \times n$. The number of columns is $n$ therefore rank $A \leq n < p$. Hence linearly dependent.

Theorem 6: The vector space $\mathbb{R}^n$ consisting of all vectors with $n$ components has dimension $n$. 
General Properties of Solutions

Fundamental Theorem for Linear Systems

(a) A linear system of \( m \) equations in \( n \) unknowns

\[
\begin{align*}
\begin{bmatrix}
    a_{11} & \cdots & a_{1n} \\
    a_{21} & \cdots & a_{2n} \\
    \vdots &       & \vdots \\
    a_{m1} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}
&= 
\begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_m
\end{bmatrix}
\end{align*}
\]

has solutions iff the coefficient matrix and the augmented matrix

\[
\begin{align*}
A &= 
\begin{bmatrix}
    a_{11} & \cdots & a_{12} \\
    \cdots &       & \cdots \\
    a_{m1} & \cdots & a_{mn}
\end{bmatrix}
and 
\tilde{A} &= 
\begin{bmatrix}
    a_{11} & \cdots & a_{12} & b_1 \\
    \cdots &       & \cdots & \cdots \\
    a_{m1} & \cdots & a_{mn} & b_m
\end{bmatrix}
\end{align*}
\]

have the same rank
General Properties of Solutions

Fundamental Theorem for Linear Systems

(b) If this rank $r$ equals $n$ then system has precisely one solution

(c) If $r < n$ the system has infinitely many solutions, all of which are obtained by determining $r$ suitable unknowns in terms of the remaining $n-r$ unknowns, to which arbitrary values can be given

(d) If solutions exist, they can all be obtained by the Gauss elimination method
General Properties of Solutions

Fundamental Theorem for Linear Systems

Proof: (a) solution iff rank $A = \text{rank } \tilde{A}$

If $A$ has a solution $x$ then $Ax = b$ or in column vectors:

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = b$$

Since $\tilde{A}$ is obtained by attaching to $A$ the additional column $b$

theorem 1 says that rank $\tilde{A}$ equals rank $A$ or rank $A+1$.

And $b$ is a linear combination of the column vectors $c_n$ (above)

Hence rank $\tilde{A}$ cannot exceed rank $A$ so rank $\tilde{A} = \text{rank } A$

Similarly if rank $\tilde{A} = \text{rank } A$ then $b$ must be a linear combination

of the column vectors of $A$

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = b$$

Which implies that $x_1 = \alpha_1, \ldots, x_n = \alpha_n$ hence a solution
General Properties of Solutions

Fundamental Theorem for Linear Systems

Proof: (b) if rank $A = r = n$ there is precisely one solution

If rank $A = r = n$ then the set $C = \{c_1, \ldots, c_n\}$ is linearly independent - theorem 1.

The representation of $b$ i.e. $c_1x_1 + c_2x_2 + \cdots + c_nx_n = b$ must be unique because otherwise:

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = c_1\tilde{x}_1 + c_2\tilde{x}_2 + \cdots + c_n\tilde{x}_n$$

would imply

$$c_1(x_1 - \tilde{x}_1) + \cdots + c_n(x_n - \tilde{x}_n) = 0$$

so that $x_1 - \tilde{x}_1 = 0, \ldots, x_n - \tilde{x}_n = 0$

Hence the solution is the same one.
General Properties of Solutions

Fundamental Theorem for Linear Systems

Proof: (c) if rank $A = r < n$ theorem 1 says there is a linearly independent set $K$ of $r$ column vectors of $A$ such that the other $n-r$ column vectors of $A$ are linear combinations of those vectors.........
Example 1

\[
\begin{align*}
3x_1 + 2x_2 + 2x_3 - 5x_4 &= 8 \\
0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\
1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1
\end{align*}
\]

\[
\Rightarrow \tilde{A} = \begin{bmatrix}
3 & 2 & 2 & -5 & 8 \\
0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\
1.2 & -0.3 & -0.3 & 2.4 & 2.1
\end{bmatrix}
\]

By Gauss elimination $\tilde{A}$ is row equivalent to;

\[
\begin{bmatrix}
3 & 2 & 2 & -5 & 8 \\
0 & 1.1 & 1.1 & -4.4 & 1.1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

$\therefore \text{rank } A = \text{rank } \tilde{A} = 2 < n=4$

We can choose $x_3$ & $x_4$ arbitrarily.
Example 2

\[
\begin{aligned}
-x_1 + x_2 + 2x_3 &= 2 \\
3x_1 - x_2 + x_3 &= 6 \\
-x_1 + 3x_2 + 4x_3 &= 4
\end{aligned}
\]

\[
\Rightarrow \tilde{A} = \begin{bmatrix}
-1 & 1 & 2 & | & 2 \\
3 & -1 & 1 & | & 6 \\
-1 & 3 & 4 & | & 4
\end{bmatrix}
\]

By Gauss elimination \( \tilde{A} \) is row equivalent to;

\[
\begin{bmatrix}
-1 & 1 & 2 & | & 2 \\
0 & 2 & 7 & | & 12 \\
0 & 0 & -5 & | & -10
\end{bmatrix}
\]

\[
\therefore \text{rank } A = \text{rank } \tilde{A} = 3 = n=3
\]

1 unique solution
Example 3

\[
\begin{align*}
3x_1 + 2x_2 + x_3 &= 3 \\
2x_1 + x_2 + x_3 &= 0 \\
6x_1 + 2x_2 + 4x_3 &= 6
\end{align*}
\]

\[\Rightarrow \tilde{A} = \begin{bmatrix}
3 & 2 & 1 \\
2 & 1 & 1 \\
6 & 2 & 4
\end{bmatrix}
\]

By Gauss elimination \(\tilde{A}\) is row equivalent to;

\[\begin{bmatrix}
3 & 2 & 1 \\
0 & -\frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0
\end{bmatrix}
\]

\[\begin{bmatrix}
3 \\
-2 \\
12
\end{bmatrix}
\]

\[\therefore \text{rank } A = 2 < \text{rank } \tilde{A} = 3\]

therefore no solution
The Homogeneous System

The system $Ax = b$ is called **homogenous** if $b=0$. Otherwise it is called **nonhomogenous**

**Theorem 2 (Homogeneous System)**

A homogeneous linear system $Ax = 0$ or

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
    &\vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{align*}
\]

always has the trivial solution $x=0$. Nontrivial solutions exist iff $\text{rank } A = r < n$, these solutions, together with $x=0$ form a vector space of dimension $n-r$.

**Proof:** (see text book) but fairly obvious from definition of vector space and general properties.
The Homogeneous System

- The vector space of all solutions is called the **null space** of \( A \) because if we multiply any \( x \) in this null space by \( A \) we get 0.

- The dimension of the null space is called the **nullity** of \( A \).

- Theorem 2 states that
  \[
  \text{rank } A + \text{nullity } A = n
  \]
  where \( n \) is the number of unknowns.

- If \( \text{rank } A = n \) then \( \text{nullity } A = 0 \) - i.e. trivial solution only.

- If \( \text{rank } A = r < n \) then \( \text{nullity } A = n-r > 0 \).
Theorems

Theorem 3: Systems with fewer equations than unknowns
A homogeneous system of linear equations with fewer equations than unknowns always has non-trivial solutions

Proof: $Ax = 0$ and $A$ is $n \times m$ ($m$ equations, $n$ unknowns)
Since rank $A \leq m$ and $m < n$ then rank $A < n$
If rank $A = r < n$ then nullity $A = n-r > 0$ and so has non-trivial roots

Theorem 4: Nonhomogeneous System. If a nonhomogeneous linear system $Ax = b$ ($\neq 0$) has solutions then all these solutions are of the form: $x = x_0 + x_h$ where $x_0$ is any fixed solution and $x_h$ runs through all the solutions of the corresponding homogeneous system $Ax=0$

Proof: Let $x$ be any solution and $x_0$ any chosen one. Then $Ax=b$ and $Ax_0=b$ and so $A(x-x_0)=Ax-Ax_0=0$. So that $x-x_0$ is a solution of the homogeneous system and $x-x_0=x_h$ in general
Inverse of a Matrix

The inverse of an $n \times n$ matrix $A = [a_{jk}]$ is denoted $A^{-1}$ and is an $n \times n$ matrix such that:

$$AA^{-1} = A^{-1}A = I$$

where $I$ is the $n \times n$ unit matrix

- If $A$ has an inverse, then $A$ is called a **nonsingular matrix**
- If $A$ has no inverse, then $A$ is called a **singular matrix**

If $A$ has an inverse, the inverse is unique

**Proof:** If both $B$ and $C$ are inverses of $A$ then $AB = I$ and $CA = I$

so that $B = IB = (CA)B = C(AB) = CI = C$
Existence of the Inverse

Theorem 1: The inverse of an $n \times n$ matrix $A$ exists iff rank $A = n$. Hence $A$ is nonsingular if rank $A = n$ and is singular if rank $A < n$.

Proof: Consider the system $Ax = B$ with the given matrix $A$ as the coefficient matrix. If the inverse exists then

$$A^{-1}Ax = x = A^{-1}b$$

This shows that $Ax = b$ has a unique solution $x$, so that $A$ must have rank $n$.

Conversely, if rank $A = n$ then $Ax = b$ has a unique solution $x$ for any $b$ and the back substitution following Gauss elimination shows that its components $x_j$ are linear combinations of those of $b$ so we can write $x = Bb$

So that $Ax = A(Bb) = (AB)b = b$ and so $AB = I$ or $B = A^{-1}$.
Determination of the Inverse

For practically determining the inverse $A^{-1}$ of a non-singular $n \times n$ matrix $A$ we can use a variant of Gauss elimination - Gauss-Jordan elimination.

Using $A$ we form the $n$ systems $Ax_1 = e_1, \ldots, Ax_n = e_n$ where $e_j$ is a column vector with the $j$th component 1 and all the others 0.

Introducing the $n \times n$ matrices $X = [x_1 \ldots x_n]$ and $I = [e_1 \ldots e_n]$ we can combine the $n$ systems into the matrix equation $AX = I$ and the $n$ augmented matrices $[A \ e_1], \ldots, [A \ e_n]$ into a single augmented matrix $\tilde{A} = [A \ I]$.

Now $AX = I$ implies $X = A^{-1}I = A^{-1}$ and to solve $AX = I$ for $X$ we can use Gauss elimination to $\tilde{A}$ to get $[U \ H]$ where $U$ is upper triangular.

The Gauss-Jordan elimination operates on $[U \ H]$ by eliminating the entries in $U$ above the diagonal giving $[I \ K]$ the augmented matrix of $IX = A^{-1}$, Thus $K = A^{-1}$.
# Gauss-Jordan Elimination

**Example:** Find the inverse $A^{-1}$ of

\[
A = \begin{bmatrix}
-1 & 1 & 2 \\
3 & -1 & 1 \\
-1 & 3 & 4 \\
\end{bmatrix}
\]

**Solution:** Gauss Elimination gives:

\[
\begin{bmatrix}
-1 & 1 & 2 & | & 1 & 0 & 0 \\
3 & -1 & 1 & | & 0 & 1 & 0 \\
-1 & 3 & 4 & | & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
-1 & 1 & 2 & | & 1 & 0 & 0 \\
0 & 2 & 7 & | & 3 & 1 & 0 \\
0 & 0 & -5 & | & -4 & -1 & 1 \\
\end{bmatrix}
\]

This is $[U \ H]$ as produced by Gauss elimination.

The additional Gauss-Jordan steps reduce $U$ to $I$ - next page.
Gauss-Jordan Elimination

\[
\begin{bmatrix}
-1 & 1 & 2 & 1 & 0 & 0 \\
0 & 2 & 7 & 3 & 1 & 0 \\
0 & 0 & -5 & -4 & -1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & -2 & -1 & 0 & 0 \\
0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\
0 & 0 & 1 & 0.8 & 0.2 & -0.2
\end{bmatrix}
\]

- row1
- 0.5 row2
- 0.2 row3

\[
\begin{bmatrix}
1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\
0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\
0 & 0 & 1 & 0.8 & 0.2 & -0.2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\
0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\
0 & 0 & 1 & 0.8 & 0.2 & -0.2
\end{bmatrix}
\]

- row1 + 2row3
- row2 - 3.5row3

The last three columns give $A^{-1}$ - check for yourself.......