Semidefinite Relaxation for a Class of Robust QCQPs:
A Verifiable Sufficient Condition for Rank-One Solutions

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Problem Statement

Problem: A robust quadratically constrained quadratic program (QCQP)

$$\begin{aligned} \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \in \mathbb{C}^N} \quad & \sum_{i=1}^{K} \|\boldsymbol{w}_i\|_2^2 \\ \text{s.t.} \quad & \max_{\|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i\|_2 \le \varepsilon_i} \sigma_i^2 + \boldsymbol{h}_i^H \left(\sum_{j \ne i} \boldsymbol{w}_j \boldsymbol{w}_j^H - \frac{1}{\gamma_i} \boldsymbol{w}_i \boldsymbol{w}_i^H \right) \boldsymbol{h}_i \le 0, \ \ i = 1, \ldots, K, \end{aligned}$$

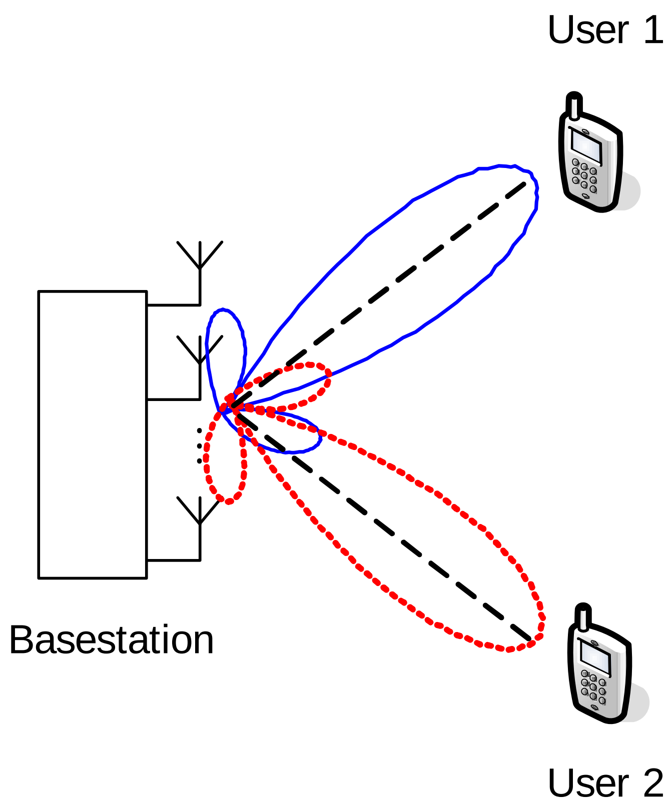
where $\bar{\boldsymbol{h}}_i \in \mathbb{C}^N$, $\sigma_i^2, \varepsilon_i, \gamma_i > 0$, $i = 1, \ldots, K$, are given.

- non-convex
- may be approximated by techniques like convex restrictions and relaxations

Question: How well does semidefinite relaxation (SDR) perform?

Motivating Application: Downlink Beamforming in Communications

Scenario: a base station (BS) sending K independent information signals to K users simultaneously; BS has N transmit antennas; users have one receive antenna.



Quality-of-service characterization: the signal-to-interference-and-noise ratios (SINRs)

$$\text{SINR}_i = \frac{|\boldsymbol{h}_i^H \boldsymbol{w}_i|^2}{\sum_{j \neq i} |\boldsymbol{h}_i^H \boldsymbol{w}_j|^2 + \sigma_i^2}, \quad i = 1, \ldots, K,$$

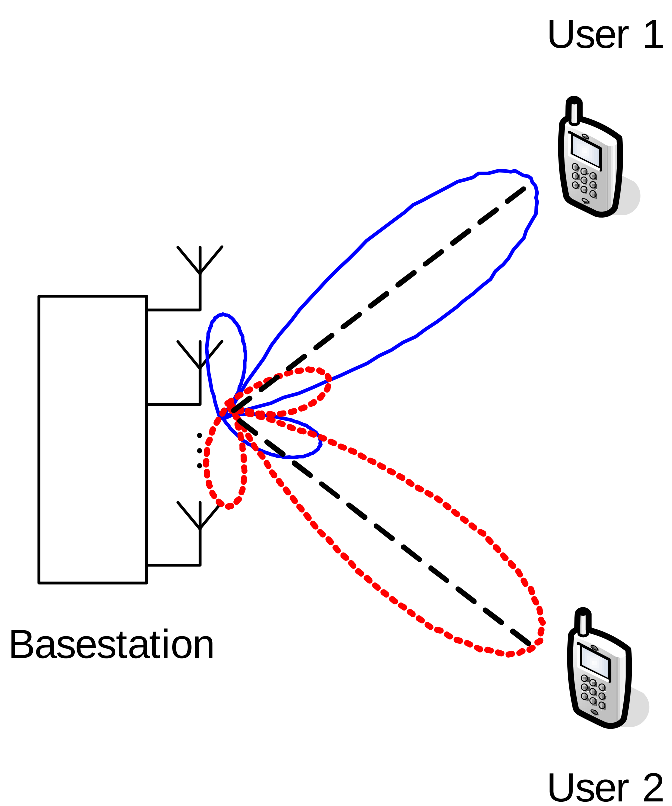
where

$\boldsymbol{h}_i \in \mathbb{C}^N$ is the channel from the BS to user i;

$\boldsymbol{w}_i \in \mathbb{C}^N$ the beamforming vector of user i;

σ_i^2 the noise power.

A Downlink Beamforming Formulation



Problem: an SINR-constrained design

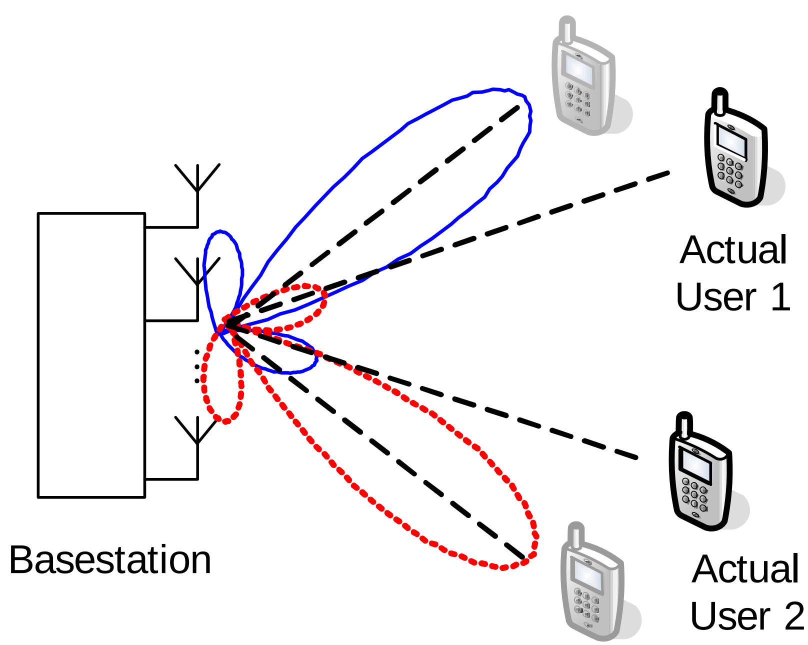
$$\begin{aligned} \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \in \mathbb{C}^N} \quad & \textstyle\sum_{i=1}^K \|\boldsymbol{w}_i\|_2^2 \\ \text{s.t.} \quad & \text{SINR}_i = \frac{|\boldsymbol{h}_i^H \boldsymbol{w}_i|^2}{\sum_{j\neq i} |\boldsymbol{h}_i^H \boldsymbol{w}_j|^2 + \sigma_i^2} \geq \gamma_i, \\ & i = 1, \ldots, K, \end{aligned}$$

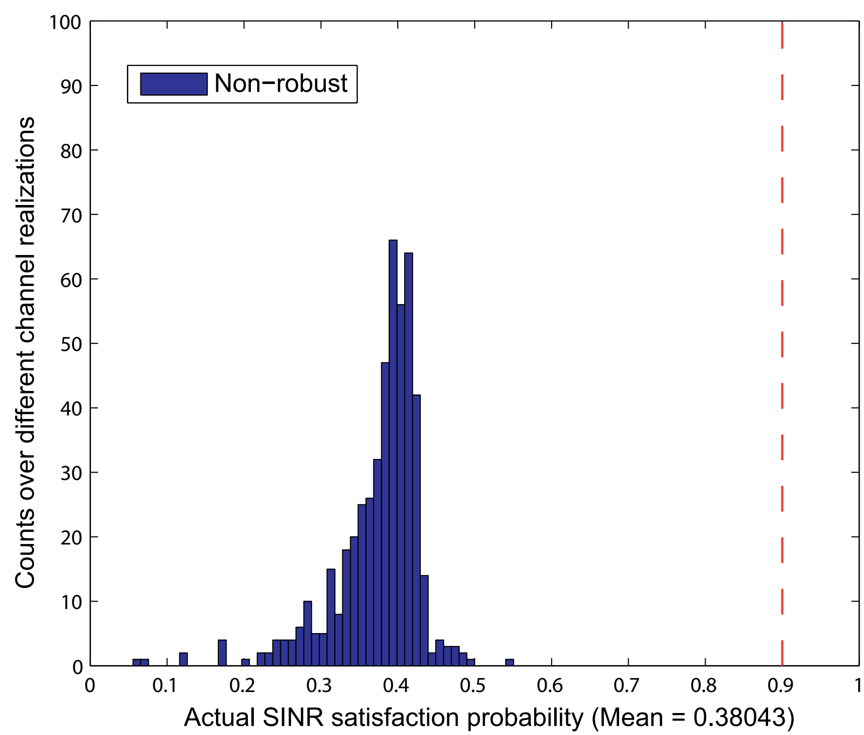
where γ_i is user-i's minimum SINR requirement.

Sensitivity Issues under Imperfect Channel Information

Issue:

- the SINR-constrained design assumes that the channels $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K$ are perfectly known at the BS;
- in practice, $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K$ are often imperfectly known

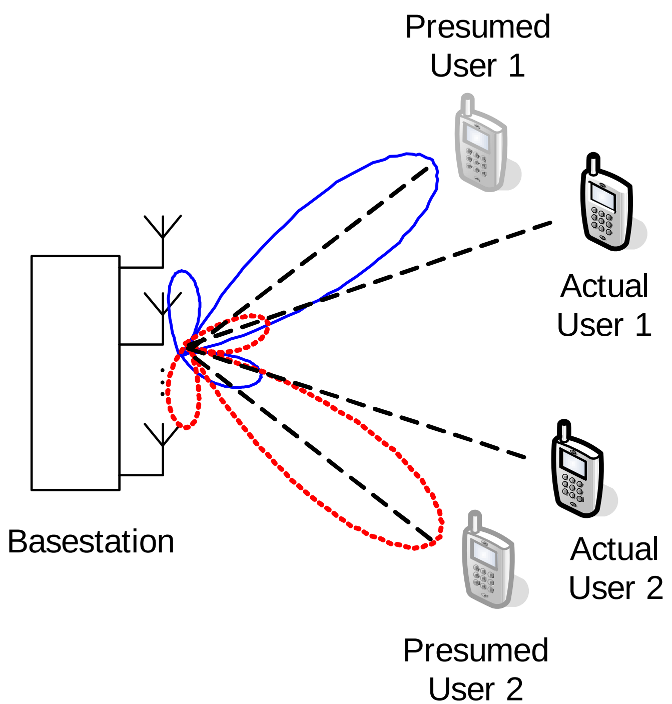




More than 50% SINR outage!

Robustifying the Beamforming Design

Goal: Guarantee that the SINR requirements are satisfied under any spherically bounded channel uncertainties.



Problem:

$$\begin{aligned} \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \in \mathbb{C}^N} \quad & \sum_{i=1}^{K} \|\boldsymbol{w}_i\|_2^2 \\ \text{s.t.} \quad & \min_{\|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i\|_2 \le \varepsilon_i} \text{SINR}_i \ge \gamma_i, \\ & \qquad i = 1, \ldots, K, \end{aligned}$$

where $\bar{\boldsymbol{h}}_i$ is the presumed channel of user i; ε_i is the uncertainty radius.

A Review of SDR: The Non-Robust Case

Recall the (non-robust) SINR-constrained design

$$\begin{aligned} \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \in \mathbb{C}^N} \quad & \textstyle\sum_{i=1}^K \|\boldsymbol{w}_i\|_2^2 \\ \text{s.t.} \quad & \mathsf{SINR}_i = \frac{|\boldsymbol{h}_i^H \boldsymbol{w}_i|^2}{\sum_{j\neq i} |\boldsymbol{h}_i^H \boldsymbol{w}_j|^2 + \sigma_i^2} \geq \gamma_i, \;\; i = 1,\ldots,K. \end{aligned}$$

SDR: apply $\boldsymbol{W}_i = \boldsymbol{w}_i \boldsymbol{w}_i^H \Longleftrightarrow \boldsymbol{W}_i \succeq \mathbf{0}$, $\mathrm{rank}(\boldsymbol{W}_i) \leq 1$ to the above problem, and drop the rank constraints to obtain a relaxed problem

$$\begin{aligned} \min_{\boldsymbol{W}_1,\ldots,\boldsymbol{W}_K \succeq \mathbf{0}} \; & \textstyle\sum_{i=1}^K \mathrm{Tr}(\boldsymbol{W}_i) \\ & \text{s.t. } \sigma_i^2 + \boldsymbol{h}_i^H \left(\textstyle\sum_{j\neq i} \boldsymbol{W}_j - \frac{1}{\gamma_i} \boldsymbol{W}_i \right) \boldsymbol{h}_i \leq 0, \;\; i = 1,\ldots,K. \end{aligned}$$

- convex, a semidefinite program (SDP)
- Question: Is SDR tight? Or, does SDR always admit a rank-one solution?

Rank-One Solution Guarantee via SDP Rank Reduction

Consider an extension of the Shapiro-Barvinok-Pataki (SBP) rank reduction result:

Fact [Huang-Palomar'09]: Consider a complex-valued SDP

$$\begin{aligned} \min_{\boldsymbol{W}_1,\ldots,\boldsymbol{W}_k \succeq \mathbf{0}} \quad & \textstyle\sum_{i=1}^{k} \mathrm{Tr}(\boldsymbol{C}_i \boldsymbol{W}_i) \\ \text{s.t.} \quad & \textstyle\sum_{l=1}^{k} \mathrm{Tr}(\boldsymbol{A}_{i,l} \boldsymbol{W}_l) \geq b_i, \quad i = 1, \ldots, m. \end{aligned}$$

If $m \leq k + 2$ and some mild assumptions hold, then there exists a solution $(\boldsymbol{W}_1^\star, \ldots, \boldsymbol{W}_k^\star)$ such that $\mathrm{rank}(\boldsymbol{W}_i^\star) = 1$ for all i.

- SDR is tight for the SINR-constrained problem since $k = m = K$
- **note:** the same conclusion can also be drawn via other proof approaches, such as uplink-downlink duality **[Bengtsson-Ottersten'01]** and a "folklore" result (to be explained).

A Review of SDR: The Robust Case

The SDR of the robust SINR-constrained design:

$$\min_{\boldsymbol{W}_i \succeq \boldsymbol{0}\ \forall i} \ \sum_{i=1}^{K} \mathrm{Tr}(\boldsymbol{W}_i) \tag{P.1}$$

$$\text{s.t.} \ \max_{\|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i\|_2 \le \varepsilon_i} \ \sigma_i^2 + \boldsymbol{h}_i^H \left(\sum_{j \neq i} \boldsymbol{W}_j - \frac{1}{\gamma_i} \boldsymbol{W}_i \right) \boldsymbol{h}_i \le 0, i = 1, \ldots, K. \tag{P.2}$$

- convex, but (P.2) are semi-infinite
- By applying the $\mathcal{S}$-lemma to (P.2), Problem (P) can be reformulated as an SDP

$$\min_{\substack{\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K \succeq \boldsymbol{0}, \\ t_1, \ldots, t_K \ge 0}} \ \sum_{i=1}^{K} \mathrm{Tr}(\boldsymbol{W}_i)$$

$$\text{s.t.} \ \begin{bmatrix} \boldsymbol{Q}_i + t_i \boldsymbol{I} & \boldsymbol{r}_i \\ \boldsymbol{r}_i^H & s_i - t_i \varepsilon_i^2 \end{bmatrix} \succeq \boldsymbol{0}, \ i = 1, \ldots, K,$$

where $\boldsymbol{Q}_i = \frac{1}{\gamma_i} \boldsymbol{W}_i - \sum_{j \neq i} \boldsymbol{W}_j$, $\boldsymbol{r}_i = \boldsymbol{Q}_i \bar{\boldsymbol{h}}_i$, $s_i = \bar{\boldsymbol{h}}_i^H \boldsymbol{Q}_i \bar{\boldsymbol{h}}_i - \sigma_i^2$.

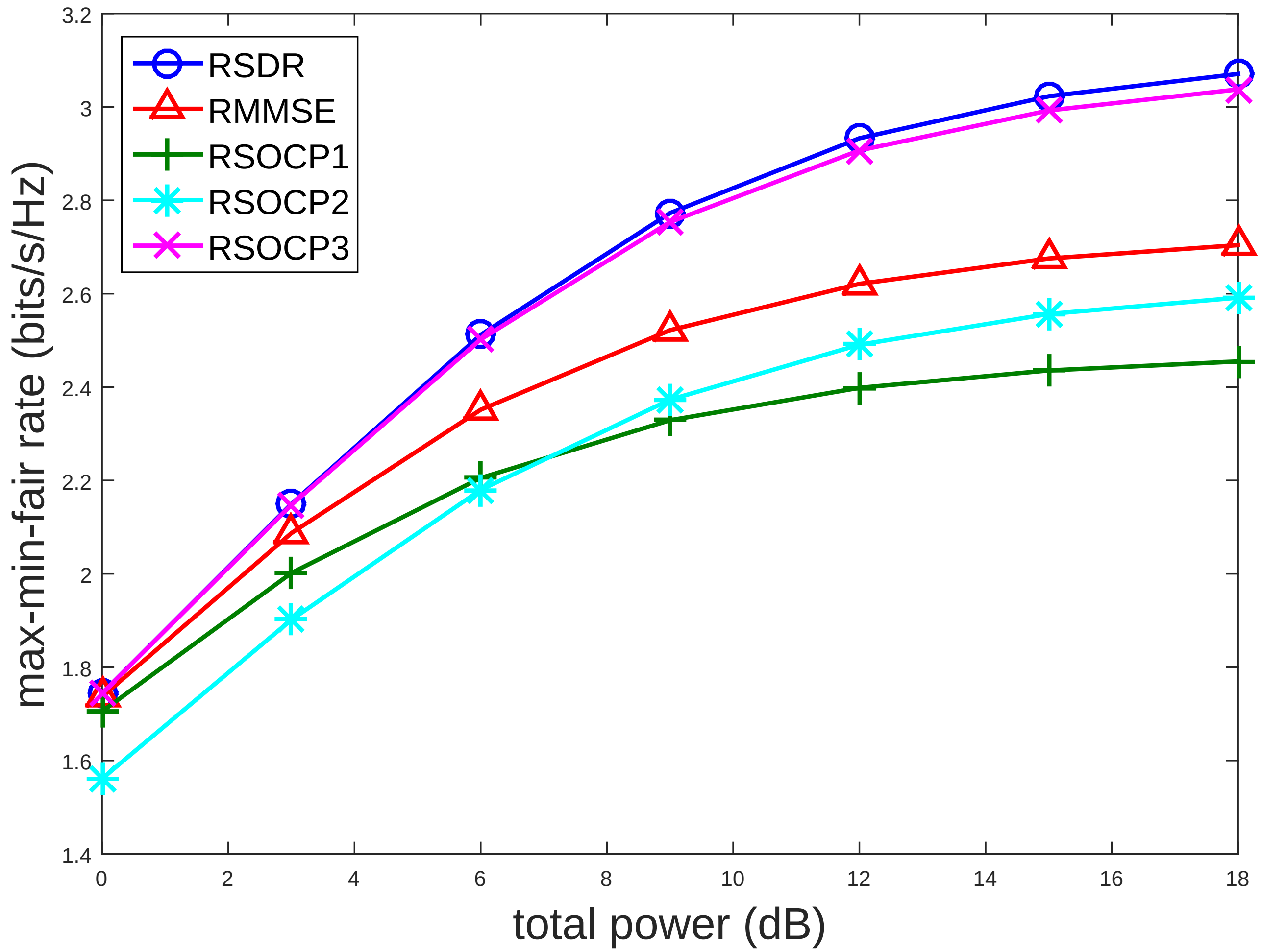
– first proposed in **[Zheng-Wang-Ng'08]**

A Curious Numerical Finding

Observation: The SDR problem was empirically found to admit a rank-one solution in almost all feasible instances!

r (bits/s/Hz)	number of rank-1 instances / number of feasible instances									
	$(N,K)=(4,3)$		$(N,K)=(8,3)$		$(N,K)=(8,7)$		$(N,K)=(12,7)$		$(N,K)=(12,11)$	
	$\varepsilon_i^2=0.1$	$\varepsilon_i^2=0.05$	$\varepsilon_i^2=0.1$	$\varepsilon_i^2=0.05$	$\varepsilon_i^2=0.1$	$\varepsilon_i^2=0.05$	$\varepsilon_i^2=0.1$	$\varepsilon_i^2=0.05$	$\varepsilon_i^2=0.1$	$\varepsilon_i^2=0.05$
0.1375	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000
0.2122	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000
0.3233	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000
0.4835	**1999/2000**	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000
0.7057	**1999/2000**	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000
1.0000	1973/1973	1995/1995	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000
1.3701	1933/1933	1993/1993	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000	2000/2000
1.8122	1688/1688	1889/1889	2000/2000	2000/2000	**1950/1952**	1997/1997	2000/2000	2000/2000	2000/2000	2000/2000
2.3165	1535/1535	1833/1833	2000/2000	2000/2000	1084/1084	1814/1814	1999/1999	2000/2000	**1483/1485**	1976/1976
2.8698	1258/1258	1743/1743	2000/2000	2000/2000	271/ 271	995/ 995	1964/1964	1998/1998	109/ 109	1068/1068
3.4594	839/ 839	1539/1539	1994/1994	2000/2000	51/ 51	549/ 549	1795/1795	1993/1993	6/ 6	160/ 160
4.0746	365/ 365	1187/1187	1961/1961	2000/2000	4/ 4	181/ 181	1262/1262	1936/1936	0/ 0	28/ 28
4.7070	68/ 68	688/ 688	1753/1753	1987/1987	0/ 0	19/ 19	354/ 354	1659/1659	0/ 0	2/ 2
5.3509	1/ 1	211/ 211	955/ 955	1920/1920	0/ 0	0/ 0	12/ 12	885/ 885	0/ 0	0/ 0
6.0022	0/ 0	21/ 21	106/ 106	1485/1485	0/ 0	0/ 0	0/ 0	122/ 122	0/ 0	0/ 0
6.6582	0/ 0	0/ 0	1/ 1	469/ 469	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0

Comparison with Other Approximation Methods



$N = 4$, $K = 3$, $\sigma_i^2 = 0.1$, $\varepsilon_i^2 = 0.1$. RSDR= robust SDR; RMMSE= **[Vučić-Boche'09]**, SOCP1= **[Shenouda-Davidson'07]**, SOCP2= **[Tajer-Prasad-Wang'11]**, SOCP3= **[Huang-Palomar-Zhang'13]**. The benchmarked methods are convex restrictions.

Our Main Interest

The SDR problem:

$$\begin{aligned} \min_{\boldsymbol{W}_i \succeq \boldsymbol{0}\ \forall i} \ & \textstyle\sum_{i=1}^{K} \mathrm{Tr}(\boldsymbol{W}_i) \\ \text{s.t.} \ & \max_{\|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i\|_2 \leq \varepsilon_i} \sigma_i^2 + \boldsymbol{h}_i^H \left(\textstyle\sum_{j \neq i} \boldsymbol{W}_j - \frac{1}{\gamma_i} \boldsymbol{W}_i \right) \boldsymbol{h}_i \leq 0, \ \ i = 1, \ldots, K. \end{aligned} \tag{P}$$

Challenge: Can we theoretically identify conditions under which Problem (P) is guaranteed to admit a rank-one solution?

Can We Call Our Old Friend, SBP Rank Reduction?

Recall the SDP form of Problem (P):

$$\min_{\substack{\boldsymbol{W}_1,\ldots,\boldsymbol{W}_K \succeq \boldsymbol{0}, \\ \boldsymbol{Z}_1,\ldots,\boldsymbol{Z}_K \succeq \boldsymbol{0}, \\ t_1,\ldots,t_K \geq 0}} \sum_{i=1}^{K} \mathrm{Tr}(\boldsymbol{W}_i) \tag{P-SDP.1}$$

$$\text{s.t. } \boldsymbol{Z}_i = \begin{bmatrix} \boldsymbol{Q}_i + t_i \boldsymbol{I} & \boldsymbol{r}_i \\ \boldsymbol{r}_i^H & s_i - t_i \varepsilon_i^2 \end{bmatrix}, \; i = 1, \ldots, K. \tag{P-SDP.2}$$

Question: Can we apply SBP rank reduction to Problem (P-SDP), just as in the non-robust case, to obtain a rank-one solution result?

- Answer: No, at least by our experience.
  - Why? Each matrix equality constraint in (P-SDP.P2) contains many scalar equality constraints.

An Existing Result by Song, Shi, Sanjabi, Sun and Luo

Denote the optimal value of Problem (P) by

$$v^\star = \min_{\boldsymbol{W}_i, \boldsymbol{Z}_i \succeq \boldsymbol{0}, t_i \geq 0, \ \forall i} \ \sum_{i=1}^K \mathrm{Tr}(\boldsymbol{W}_i)$$

$$\text{s.t. } \boldsymbol{Z}_i = \begin{bmatrix} \boldsymbol{Q}_i + t_i \boldsymbol{I} & \boldsymbol{r}_i \\ \boldsymbol{r}_i^H & s_i - t_i \varepsilon_i^2 \end{bmatrix}, \ i = 1, \ldots, K.$$

Result [Song-Shi-Sanjabi-Sun-Luo'12]: A solution $(\boldsymbol{W}_1^\star, \ldots, \boldsymbol{W}_K^\star)$ to Problem (P) must be of rank one if

$$\varepsilon_i^2 < \frac{\gamma_i \sigma_i^2}{v^\star}, \text{ for } i = 1, \ldots, K.$$

Implication: Problem (P) should admit a rank-one solution under sufficiently small error bounds ε_i's.

Drawback: unverifiable; $v^\star$ also depends on the problem instance $\{\bar{\boldsymbol{h}}_i, \sigma_i^2, \varepsilon_i, \gamma_i\}_{i=1}^K$.

A Verifiable Result by Us

Let

$$\hat{\boldsymbol{F}} = [\ \bar{\boldsymbol{h}}_1/\|\bar{\boldsymbol{h}}_1\|_2, \ldots, \bar{\boldsymbol{h}}_K/\|\bar{\boldsymbol{h}}_K\|_2\ ]$$

be the presumed multiuser channel direction matrix.

Result [Ma-Pan-So-Chang'16]: Under a few mild assumptions, a solution $(\boldsymbol{W}_i^\star)_{i=1}^K$ to Problem (P) must be of rank one if

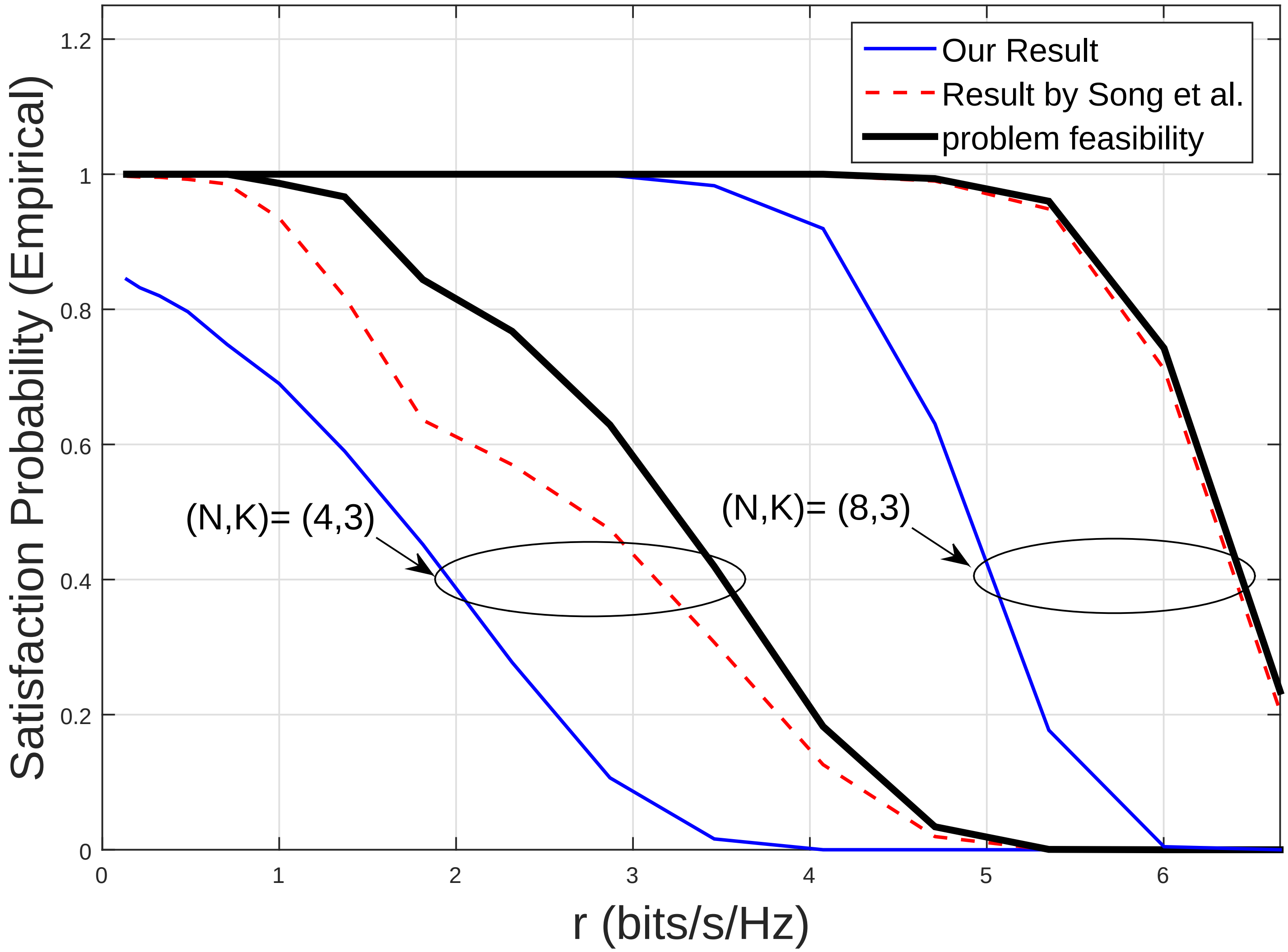
$$\frac{\|\bar{\boldsymbol{h}}_k\|_2^2}{\varepsilon_k^2}\sigma_{\min}(\hat{\boldsymbol{F}})^2 > 1 + K + \left(K - \frac{1}{K}\right)\gamma_k, \quad k = 1, \ldots, K,$$

where $\sigma_{\min}(\hat{\boldsymbol{F}})$ is the smallest singular value of $\hat{\boldsymbol{F}}$.

Implication: The SDR problem will admit a rank-one solution if

- the channel-to-uncertainty ratios $\|\bar{\boldsymbol{h}}_k\|_2^2/\varepsilon_k^2$ are sufficiently large;
- the channel direction matrix $\hat{\boldsymbol{F}}$ is sufficiently well-conditioned.

Tightness of Our Verifiable Condition



There is a gap between our verifiable condition and numerical result. Nevertheless, the performance trends of the two are consistent.

Proof Sketch of Our Result: Setting the Stage

Let us write the robust SDR problem as

$$\begin{aligned} \min_{\mathcal{W} \in \mathcal{S}} \; & \textstyle\sum_{i=1}^{K} \mathrm{Tr}(\boldsymbol{W}_i) \\ \text{s.t.} \; & \max_{\|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i\|_2 \le \varepsilon_i} \varphi_i(\mathcal{W}, \boldsymbol{h}_i) \le 0, \; i = 1, \ldots, K. \end{aligned}$$

where $\mathcal{W} = (\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K)$, $\mathcal{S} = \{ \mathcal{W} \mid \boldsymbol{W}_i \succeq \mathbf{0} \; \forall i \}$,

$$\varphi_i(\mathcal{W}, \boldsymbol{h}_i) = \sigma_i^2 + \boldsymbol{h}_i^H \left(\textstyle\sum_{j \neq i} \boldsymbol{W}_j - \frac{1}{\gamma_i} \boldsymbol{W}_i \right) \boldsymbol{h}_i.$$

- φ_i is affine in $\mathcal{W}$ and indefinite in $\boldsymbol{h}_i$

Proof Sketch of Our Result: An Equivalent Representation of the Robust Constraints

Let's do SDR with the robust constraint functions:

$$\begin{aligned}
&\max_{\|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i\|_2 \le \varepsilon_i} \varphi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{h}_i) \\
&= \max_{\|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i\|_2 \le \varepsilon_i} \sigma_i^2 + \mathrm{Tr}\left(\boldsymbol{h}_i \boldsymbol{h}_i^H \left(\textstyle\sum_{j \ne i} \boldsymbol{W}_j - \frac{1}{\gamma_i} \boldsymbol{W}_i \right) \right) \\
&\le \max_{\boldsymbol{H}_i \in \mathcal{V}_i} \sigma_i^2 + \mathrm{Tr}\left(\boldsymbol{H}_i \left(\textstyle\sum_{j \ne i} \boldsymbol{W}_j - \frac{1}{\gamma_i} \boldsymbol{W}_i \right) \right) \triangleq \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i)
\end{aligned}$$

where $\mathcal{V}_i = \{ \boldsymbol{H}_i \mid \exists \boldsymbol{h}_i \text{ s.t. } \boldsymbol{H}_i \succeq \boldsymbol{h}_i \boldsymbol{h}_i^H, \|\bar{\boldsymbol{h}}_i\|_2^2 - 2\mathrm{Re}(\bar{\boldsymbol{h}}_i^H \boldsymbol{h}_i) + \mathrm{Tr}(\boldsymbol{H}_i) \le \varepsilon_i^2 \}$.

SDR is tight in this case (SBP rank reduction). Thus,

$$\max_{\|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i\|_2 \le \varepsilon_i} \varphi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{h}_i) = \max_{\boldsymbol{H}_i \in \mathcal{V}_i} \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i).$$

- ϕ_i is affine in $\boldsymbol{\mathcal{W}}$ and affine in $\boldsymbol{H}_i$.

Proof Sketch of Our Result: A New Duality Result

Theorem: Under a few mild assumptions, we have

$$\begin{aligned} \min_{\boldsymbol{\mathcal{W}} \in \mathcal{S}} \ & \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) \\ \text{s.t.} \ & \max_{\boldsymbol{H}_i \in \mathcal{V}_i} \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) \leq 0, \ \forall i \end{aligned} \quad = \quad \begin{aligned} \max_{\boldsymbol{\mathcal{H}}_i \in \mathcal{V}_i \ \forall i} \ \min_{\boldsymbol{\mathcal{W}} \in \mathcal{S}} \ & \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) \\ \text{s.t.} \ & \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) \leq 0, \ \forall i \end{aligned}$$

Also,

$\boldsymbol{\mathcal{W}}^\star$ is a solution to the LHS problem $\Longrightarrow$ there exists $\boldsymbol{\mathcal{H}}^\star$ such that $(\boldsymbol{\mathcal{H}}^\star, \boldsymbol{\mathcal{W}}^\star)$ is a maximin solution to the RHS problem

- Proof Idea: Sion's maximin theorem and some simple arguments; the affine property of ϕ_i is crucial.

Proof Sketch of The New Duality Theorem

$$
\begin{aligned}
\text{LHS Problem} &= \min_{\boldsymbol{W}\in\bar{\mathcal{S}}} \sup_{\boldsymbol{\lambda}\geq\mathbf{0}} \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) + \sum_i \lambda_i \sup_{\boldsymbol{H}_i\in\mathcal{V}_i} \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) \\
&= \sup_{\boldsymbol{\lambda}\geq\mathbf{0}} \min_{\boldsymbol{W}\in\bar{\mathcal{S}}} \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) + \sum_i \lambda_i \sup_{\boldsymbol{H}_i\in\mathcal{V}_i} \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) && \text{(a)} \\
&= \sup_{\boldsymbol{\lambda}\geq\mathbf{0}} \min_{\boldsymbol{W}\in\bar{\mathcal{S}}} \sup_{\boldsymbol{H}_i\in\mathcal{V}_i\ \forall i} \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) + \sum_i \lambda_i \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) \\
&= \sup_{\boldsymbol{\lambda}\geq\mathbf{0}} \sup_{\boldsymbol{H}_i\in\mathcal{V}_i\ \forall i} \min_{\boldsymbol{W}\in\bar{\mathcal{S}}} \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) + \sum_i \lambda_i \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) && \text{(b)} \\
&= \sup_{\boldsymbol{H}_i\in\mathcal{V}_i\ \forall i} \sup_{\boldsymbol{\lambda}\geq\mathbf{0}} \min_{\boldsymbol{W}\in\bar{\mathcal{S}}} \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) + \sum_i \lambda_i \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) \\
&= \sup_{\boldsymbol{H}_i\in\mathcal{V}_i\ \forall i} \min_{\boldsymbol{W}\in\bar{\mathcal{S}}} \sup_{\boldsymbol{\lambda}\geq\mathbf{0}} \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) + \sum_i \lambda_i \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) && \text{(c)} \\
&= \text{RHS Problem}
\end{aligned}
$$

where (a), (b) and (c) are all due to Sion's maximin theorem.

- all about flipping min and sup!
- the affine property of ϕ_i is essential in (b).

Proof Sketch of Our Result: Further Discussion

Theorem: Under a few mild assumptions, we have

$$\begin{aligned} &\min_{\boldsymbol{\mathcal{W}} \in \mathcal{S}} \ \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) \\ &\ \text{s.t.} \ \max_{\boldsymbol{H}_i \in \mathcal{V}_i} \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) \leq 0, \ \forall i \end{aligned} \quad = \quad \begin{aligned} &\max_{\boldsymbol{\mathcal{H}}_i \in \mathcal{V}_i \ \forall i} \ \min_{\boldsymbol{\mathcal{W}} \in \mathcal{S}} \ \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) \\ &\qquad\qquad \text{s.t.} \ \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) \leq 0, \ \forall i \end{aligned}$$

Also,

$\boldsymbol{\mathcal{W}}^\star$ is a solution to the LHS problem $\Longrightarrow$ there exists $\boldsymbol{\mathcal{H}}^\star$ such that $(\boldsymbol{\mathcal{H}}^\star, \boldsymbol{\mathcal{W}}^\star)$ is a maximin solution to the RHS problem

Discussion:

- every inner problem on the RHS has a rank-one solution (SBP rank reduction)
- does that imply that the LHS problem has a rank-one solution?

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Discussion:

- every inner problem on the RHS has a rank-one solution (SBP rank reduction)
- does that imply that the LHS problem has a rank-one solution?
  - No, the theorem didn't say

$(\boldsymbol{\mathcal{H}}^\star, \boldsymbol{\mathcal{W}}^\star)$ is a maximin solution to the RHS problem $\implies$ $\boldsymbol{\mathcal{W}}^\star$ is a solution to the LHS problem

Proof Sketch of Our Result: Further Discussion

Theorem: Under a few mild assumptions, we have

$$\begin{aligned} \min_{\mathcal{W}\in\mathcal{S}} \ & \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) \\ \text{s.t.} \ & \max_{\boldsymbol{H}_i\in\mathcal{V}_i} \phi_i(\mathcal{W}, \boldsymbol{H}_i) \le 0, \ \forall i \end{aligned} \quad = \max_{\mathcal{H}_i\in\mathcal{V}_i \ \forall i} \ \begin{aligned} \min_{\mathcal{W}\in\mathcal{S}} \ & \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) \\ \text{s.t.} \ & \phi_i(\mathcal{W}, \boldsymbol{H}_i) \le 0, \ \forall i \end{aligned}$$

Also,

$\mathcal{W}^\star$ is a solution to the LHS problem $\Longrightarrow$ there exists $\mathcal{H}^\star$ such that $(\mathcal{H}^\star, \mathcal{W}^\star)$ is a maximin solution to the RHS problem

Discussion:

- however, if every inner problem on the RHS must admit a rank-one solution, then the solution $\mathcal{W}^\star$ to the LHS problem must be of rank one.
  - why don't we check when such instances happen?

Proof Sketch of Our Result: A Different Rank-One Result

Consider the non-robust SINR-constrained design

$$\begin{aligned} \min_{\boldsymbol{\mathcal{W}} \in \mathcal{S}} \ & \textstyle\sum_i \mathrm{Tr}(\boldsymbol{W}_i) \\ \text{s.t.} \ & \phi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{H}_i) = \sigma_i^2 + \mathrm{Tr}\left(\boldsymbol{H}_i \left(\textstyle\sum_{j \neq i} \boldsymbol{W}_j - \frac{1}{\gamma_i}\boldsymbol{W}_i\right)\right) \leq 0, \ i = 1, \ldots, K \end{aligned} \tag{P2}$$

where $\boldsymbol{H}_i \in \mathcal{V}_i$ for all i.

Aim: Identify conditions under which a solution to (P2) must have rank one.

- SBP rank reduction doesn't work; it's only good at saying "there exists"

Proof Sketch of Our Result: A Different Rank-One Result

Consider the non-robust SINR-constrained design

$$\min_{\mathcal{W} \in \mathcal{S}} \ \sum_i \mathrm{Tr}(\boldsymbol{W}_i)$$

$$\text{s.t.} \ \ \phi_i(\mathcal{W}, \boldsymbol{H}_i) = \sigma_i^2 + \mathrm{Tr}\left(\boldsymbol{H}_i \left(\sum_{j \neq i} \boldsymbol{W}_j - \frac{1}{\gamma_i}\boldsymbol{W}_i\right)\right) \leq 0, \ \ i = 1, \ldots, K \qquad \text{(P2)}$$

where $\boldsymbol{H}_i \in \mathcal{V}_i$ for all i.

Fact (folklore): If all $\boldsymbol{H}_i$'s take a rank-one form $\boldsymbol{H}_i = \boldsymbol{h}_i \boldsymbol{h}_i^H$, then a solution to (P2) must have rank one.

- Proof Idea: exploit the specific structures of the dual of (P2). Particularly, the dual variables of (P2) w.r.t. $\boldsymbol{W}_i$'s take the form

$$\boldsymbol{Z}_i = \boldsymbol{I} + \sum_{j \neq i} \mu_j \boldsymbol{H}_j - \frac{\mu_i}{\gamma_i}\boldsymbol{H}_i \succeq \boldsymbol{0}, \ \ \boldsymbol{\mu} \geq \boldsymbol{0} \Longrightarrow \mathrm{rank}(\boldsymbol{Z}_i) \geq N - 1$$

  The complementary slackness $\boldsymbol{Z}_i \boldsymbol{W}_i = \boldsymbol{0}$ enforces $\mathrm{rank}(\boldsymbol{W}_i) \leq 1$.

Proof Sketch of Our Result: A Different Rank-One Result

Consider the non-robust SINR-constrained design

$$\min_{\mathcal{W}\in\mathcal{S}} \ \sum_i \mathrm{Tr}(\boldsymbol{W}_i)$$
$$\text{s.t.}\ \ \phi_i(\mathcal{W}, \boldsymbol{H}_i) = \sigma_i^2 + \mathrm{Tr}\left(\boldsymbol{H}_i\left(\sum_{j\neq i}\boldsymbol{W}_j - \frac{1}{\gamma_i}\boldsymbol{W}_i\right)\right) \leq 0,\ \ i=1,\ldots,K \tag{P2}$$

where $\boldsymbol{H}_i \in \mathcal{V}_i$ for all i.

Fact (folklore): If all $\boldsymbol{H}_i$'s take a rank-one form $\boldsymbol{H}_i = \boldsymbol{h}_i\boldsymbol{h}_i^H$, then a solution to (P2) must have rank one.

Our Finishing Touch:

- every $\boldsymbol{H}_i \in \mathcal{V}_i$ can be written as $\boldsymbol{H}_i = \boldsymbol{h}_i\boldsymbol{h}_i^H + \boldsymbol{\Xi}_i$ for some $\boldsymbol{h}_i$, $\boldsymbol{\Xi} \succeq \boldsymbol{0}$;
- study a variation of the folklore fact for $\boldsymbol{H}_i = \boldsymbol{h}_i\boldsymbol{h}_i^H + \boldsymbol{\Xi}_i$ (with $\boldsymbol{\Xi}_i$ being small);
- identify conditions under which (P2) must have rank-one solutions for all $\boldsymbol{H}_i \in \mathcal{V}_i$

Conclusion and Discussion

- We considered a specific robust QCQP and showed a verifiable sufficient condition under which SDR is tight.

- Future challenge: Can we establish a strong rank-one solution result? Simulation results indicate the SDR solution is almost always of rank one.

Thank you. Preprint available on https://arxiv.org/abs/1602.01569 or

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