Quasi-Maximum-Likelihood Multiuser Detection using Semi-Definite Relaxation with Application to Synchronous CDMA

Revised Version

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Abstract

The maximum-likelihood (ML) multiuser detector is well known to exhibit better bit-error-rate (BER) performance than many other multiuser detectors. Unfortunately, ML detection (MLD) is a non-deterministic polynomial-time hard (NP-hard) problem, for which there is no known algorithm that can find the optimal solution with polynomial-time complexity (in the number of users). In this paper, a polynomial-time approximation method called semi-definite (SD) relaxation is applied to the MLD problem with anti-podal data transmission. SD relaxation is an accurate approximation method for certain NP-hard problems. The SD relaxation ML (SDR-ML) detector is efficient in that its complexity is of the order of $K^{3.5}$, where K is the number of users. We illustrate the potential of the SDR-ML detector by showing that some existing detectors such as the decorrelator and the linear-minimum-mean-square-error detector can be interpreted as degenerate forms of the SDR-ML detector. Simulation results indicate that the BER performance of the SDR-ML detector is better than that of these existing detectors and is close to that of the true ML detector, even when the cross-correlations between users are strong or the near-far effect is significant.

KEYWORDS—maximum likelihood detection, multiuser detection, relaxation methods, semi-definite programming

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1 Introduction

In code division multiple access (CDMA), a major factor that limits system performance is the multiuser interference caused by the non-orthogonality of the user signature waveforms. Multiuser detection [1] is a powerful tool for combating the effects of this multiuser interference. Under some standard assumptions, the maximum-likelihood (ML) multiuser detector is optimum in the sense that it provides the minimum error probability in jointly detecting the data symbols of all users. Unfortunately, to implement the ML detector, it is necessary to solve a difficult combinatorial optimization problem. The ML detection (MLD) problem can be solved by an exhaustive search, in which the log likelihood function is evaluated for all possible combinations of the data symbols. However, exhaustive search is prohibitive for large number of users because of its exponentially increasing computational complexity. Hence, it would be desirable if there were algorithms that could efficiently find the (globally-optimal) MLD solution. It is known that such MLD algorithms exist if the signal correlation matrix exhibits some special structure. Two structural constraints that are known to lead to efficient MLD solutions are: (i) band-diagonal signal correlation matrix with small number of non-zero diagonals [1, 2, 3]; and (ii) non-positive cross-correlation between all pairs of signature waveforms [4]. However, these structural constraints are rather restrictive and are satisfied only in some special scenarios of multiuser communications. In fact, for an arbitrary signal correlation matrix, it is unlikely that an efficient MLD algorithm exists because the MLD problem in this general case has been shown to be a non-deterministic polynomial-time hard (NP-hard) problem [5], which implies that there is no known algorithm which can solve the MLD problem with polynomial complexity in the number of users.

Because of the intrinsic difficulty in solving the MLD problem, there has been much interest in the development of suboptimal but computationally efficient ML detectors. A tree search method has been proposed to perform an incomplete search for the MLD solution with limited complexity [6]. Multistage detection [7], the coordinate ascent algorithm [8], and the expectation-maximization (EM) approach [9] are methods in which the detected symbols are iteratively updated in an attempt to increase the log likelihood function. These iterative methods are similar in that they perform some form of interference cancellation. The drawback of interference cancellation is that if the estimation of the interference is incorrect, the interference may be aggravated by the interference-canceling operation.

In this paper, we examine the approximation of the MLD solution using relaxation methods. Relaxation is an effective approximation technique for certain difficult optimization problems. Its rationale is simple in that it relaxes some of the constraints of the optimization problem such that the relaxed problem is easier to solve than the original problem. In this work, most of our emphasis will be placed on semi-definite (SD) relaxation [10, 11], which is an accurate and efficient approximation method for certain kinds of NP-hard problems. We will describe the SD relaxation algorithm for the Boolean quadratic-programming (QP) problem [11] and will show how this algorithm can be applied to the MLD problem with anti-podal data transmission. There are three advantages of employing SD relaxation:

1. The SD relaxation algorithm is based on solving a convex optimization problem. Hence, this method does not suffer from local maxima.

2. The relaxed problem is a semi-definite programming problem, which is known to be efficiently solvable [12].

3. The SD relaxation algorithm has a theoretical guarantee that the approximation accuracy is, at worst, moderate [11]. Moreover, the performance of this algorithm in practice is substantially better than that of the worst case.

In addition to SD relaxation, we also consider two other relaxation methods which have recently been applied to multiuser detection [13, 14], namely unconstrained relaxation and bound relaxation. These two relaxation algorithms are related to some existing detectors, such as the well-known decorrelating and linear-minimum-mean-square-error (LMMSE) detectors, and a modified form of the space-alternating generalized expectation-maximization (SAGE) detector [9]. We will show that both the unconstrained and bound relaxation methods can be considered as further relaxations of the SD relaxation method. This result suggests that the SD relaxation ML (SDR-ML) detector should perform better than those existing detectors. This viewpoint will be supported by our simulation results for synchronous CDMA, where the bit error rate (BER) performance of the SDR-ML detector is shown to be better than that of other suboptimal detectors including the decorrelator, the LMMSE detector, and the SAGE detectors. Our simulation results will also show that the SDR-ML detector yields a BER performance close to that of the true ML detector, even when the cross-correlations between users are strong or the near-far effect is significant.

2 Problem Statement

Assuming anti-podal data transmission, the received signal for a multiuser communication system can be represented by the following equation:

$$r(t) = \sum_{k=1}^{K} \sqrt{E_k} b_k s_k(t) + v(t), \quad t \in \mathcal{T} \tag{2.1}$$

where $b_k \in \{-1, +1\}$ is the information symbol transmitted by kth user, $s_k(t)$ is unit-energy waveform carrying the information symbol of the kth user, E_k is the received signal energy for the kth transmitted waveform, $v(t)$ is a zero-mean additive white Gaussian noise (AWGN) with two-sided power spectral density $\mathcal{N}_o/2$, K is the number of users, and $\mathcal{T}$ is the observation interval. We assume that $\{s_k(t)\}$ is a set of non-orthogonal signals. Eq. (2.1) can be considered as a generic model for many multiple signal detection problems, such as the CDMA multiuser detection problems [2, 1]. In particular, for the detection problem of synchronous CDMA signals over one symbol period, $\{s_k(t)\}$ is defined to be a set of symbol-synchronous spreading-code waveforms

$$s_k(t) = \sum_{i=0}^{P-1} c_{ki} \psi(t - iT_p) \tag{2.2}$$

where T_b is the symbol duration, P is the number of chips, $T_p = T_b/P$ is the chip duration, $\{c_{ki}\}_{i=0}^{P-1}$ is the spreading-code sequence for the kth user, and $\psi(t)$ is a self-orthogonal chip waveform over the interval T_p.

The purpose of multiuser detection is to detect $\{b_k\}$ given the observed signal $r(t)$. The ML detector for such a received signal is optimum in that the probability of incorrectly detecting $\mathbf{b} = [\ b_1, b_2, \ldots, b_K\ ]^T$ is minimized, under the usual assumptions that: (i) the sets $\{s_k(t)\}$ and $\{E_k\}$ are known; and (ii) the bits b_k are independent and identically distributed (i.i.d.) and equiprobable.[1] Maximum likelihood detection (MLD) can be formulated as [1]

$$\hat{\mathbf{b}}_{\text{ML}} = \arg \max_{\mathbf{b} \in \{-1,+1\}^K} J(\mathbf{b}) \tag{2.3}$$

where

$$J(\mathbf{b}) = 2\mathbf{b}^T \mathbf{A} \mathbf{y} - \mathbf{b}^T \mathbf{H} \mathbf{b} \tag{2.4}$$

[1] An additional property of the ML detector is that for sufficiently high signal-to-noise ratios, its bit error probability for any individual b_k approaches the lowest achievable [1],[15, pp. 45-48].

is an objective function proportional to the log likelihood function. Here, the kth element of the vector $\mathbf{y}$ is given by

$$y_k = \int_{\mathcal{T}} r(t)s_k(t)dt \tag{2.5}$$

and is the sampled matched filter output for the kth user, and $\mathbf{H} = \mathbf{ARA}$ is the signal correlation matrix with $\mathbf{A} = \mathrm{diag}(\sqrt{E}_1, \ldots, \sqrt{E}_K)$ and

$$R_{ij} = \int_{\mathcal{T}} s_i(t)s_j(t)dt \tag{2.6}$$

being the (i, j)th element of normalized correlation matrix $\mathbf{R}$. To perform MLD, it is necessary to solve the combinatorial optimization problem in (2.3). As pointed out in Section 1, Problem (2.3) can be solved using an exhaustive search, in which the MLD solution is found by evaluating $J(\mathbf{b})$ for all $\mathbf{b} \in \{-1, 1\}^K$. However, the complexity of the exhaustive search is $\mathcal{O}(2^K)$ and thus is prohibitive for large K. In fact, for an arbitrary $\mathbf{H}$, it is unlikely that the MLD problem can be efficiently solved (in the sense of polynomial-time complexity in K) because Problem (2.3) is an NP-hard problem [5].

3 Some Basic Concepts of Relaxation

In general, an optimization problem can be formulated as

$$f^\star = \max_{\mathbf{x} \in \mathcal{S}} f(\mathbf{x}) \tag{3.1}$$

where $\mathcal{S}$, $f(\mathbf{x})$ and $f^\star$ represent the feasible set, the objective function and the maximum objective value achieved by (3.1), respectively. The following problem

$$g^\star = \max_{\mathbf{x} \in \mathcal{U}} f(\mathbf{x}) \tag{3.2}$$

is called a relaxation of (3.1) if $\mathcal{S} \subset \mathcal{U}$. The intent of relaxation is to make the relaxed problem easier to solve than the original problem by appropriately choosing $\mathcal{U}$. However, the solution to the relaxed problem cannot be directly used as an approximate solution to (3.1) because it may not lie in $\mathcal{S}$. For this reason, some approximation techniques are usually required to convert the relaxation solution to an approximate solution to the original problem. Hence, a relaxation algorithm should consist of two steps: (i) solve the relaxed problem; and (ii) use an approximation algorithm to convert the relaxation solution to an approximate solution to the original problem.

We now illustrate the principle of relaxation by a simple example in which an unconstrained relaxation method [14, 13] is applied to the MLD problem. For simplicity, we assume $\mathbf{R}$ to be of full rank. Now, if the alphabet constraint in (2.3) is removed, we obtain the following relaxation:

$$\max_{\mathbf{b} \in \mathbb{R}^K} J(\mathbf{b}) \tag{3.3}$$

Problem (3.3) is a least-squares problem and thus is much easier to solve than the MLD problem. It can be shown that the solution to (3.3) is $\mathbf{H}^{-1}\mathbf{A}\mathbf{y}$ [16]. Denote the approximate MLD solution by $\hat{\mathbf{b}}_{\mathrm{UR}}$. To obtain $\hat{\mathbf{b}}_{\mathrm{UR}}$, we can apply an element-by-element threshold decision to the solution to (3.3); i.e., choose $\hat{b}_{\mathrm{UR},k} = -1$ if $[\mathbf{H}^{-1}\mathbf{A}\mathbf{y}]_k < 0$, and $\hat{b}_{\mathrm{UR},k} = 1$ otherwise. In fact, this relaxation method is closely related to one commonly-known suboptimal detector, namely the decorrelator. This will be elaborated in Section 5.

We see that the maximum objective value achieved by the relaxed problem provides an upper bound on the maximum objective value achieved by the original problem; i.e., $f^\star \leq g^\star$. Hence, if the gap $g^\star - f^\star$ is reduced, it is possible that the relaxation solution can be made closer to the solution of the original problem and a more accurate approximate solution may be obtained. Similarly, if we have another relaxation

$$h^\star = \max_{\mathbf{x} \in \mathcal{V}} f(\mathbf{x}), \qquad \mathcal{S} \subset \mathcal{V} \tag{3.4}$$

and $h^\star$ provides a tighter upper bound than $g^\star$ (i.e., $h^\star \leq g^\star$), then it is expected that relaxation (3.4) should lead to a better approximation than relaxation (3.2). We say that one relaxation is tighter than another if the maximum objective value of the former is no greater than that of the latter. In designing a relaxation algorithm with good approximation accuracy, it is beneficial to employ a tight relaxation.

4 Application of SD Relaxation to MLD

Unlike the unconstrained relaxation method which modifies the alphabet constraint in the MLD problem, the SD relaxation method considers an increase in problem dimensionality so as to provide a tight relaxation. The relative tightness of various relaxation methods will be discussed in Section 5. In this section, the SD relaxation algorithm for the Boolean quadratic-programming (QP) problem will be described. Then, by showing the link between the Boolean QP problem and the MLD problem, we will describe how the SD relaxation algorithm can be applied to the MLD problem. A summary of the SDR-ML detector is provided in Table 1.

4.1 Semi-Definite Relaxation

We consider the following *Boolean QP problem*:

$$\max_{\mathbf{x}\in\{-1,+1\}^n} \mathbf{x}^T\mathbf{Q}\mathbf{x} \tag{4.1}$$

where $\mathbf{Q}$ can be any symmetric matrix. To present SD relaxation, we consider a reformulation of the Boolean QP problem. Since $\mathbf{x}^T\mathbf{Q}\mathbf{x} = \mathrm{Trace}(\mathbf{x}\mathbf{x}^T\mathbf{Q})$, Problem (4.1) is equivalent to the following problem:

$$\begin{aligned} \max \quad & \mathrm{Trace}(\mathbf{X}\mathbf{Q}) \\ \text{s.t.} \quad & \mathbf{X} = \mathbf{x}\mathbf{x}^T, \qquad \mathbf{x}\in\mathbb{R}^n \\ & X_{ii} = 1, \qquad i = 1,\ldots,n \end{aligned} \tag{4.2}$$

The constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ implies that $\mathbf{X}$ is symmetric, positive semi-definite (PSD) and of rank 1. Due to the constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, Problem (4.2) is a non-convex optimization problem. Now, if the rank-1 constraint is removed from (4.2), we obtain the following relaxed problem:

$$\begin{aligned} \max \quad & \mathrm{Trace}(\mathbf{X}\mathbf{Q}) \\ \text{s.t.} \quad & \mathbf{X} \succeq \mathbf{0} \\ & X_{ii} = 1, \qquad i = 1,\ldots,n \end{aligned} \tag{4.3}$$

where $\mathbf{X} \succeq \mathbf{0}$ means that $\mathbf{X}$ is symmetric and PSD. Problem (4.3) is known as a *semi-definite programming* (SDP) *problem* [12], and therefore (4.3) is called an *SD relaxation* of (4.1). An advantage of using SD relaxation is that (4.3) is a convex optimization problem and hence does not suffer from local maxima. Furthermore, an efficient optimization algorithm based on interior-point methods has been developed for the SDP problem in (4.3) [17, 18]. This algorithm is efficient in that for a given accuracy, a solution to (4.3) can be found in at most $\mathcal{O}(n^{3.5})$ operations.[2] The SD relaxation problem has an interesting property which will become important later:

Property 1 [11] *The solution to (4.3) is independent of the diagonal elements of* $\mathbf{Q}$.

This property can be verified by defining $\mathcal{P} = \{\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n\times n} : P_{ii} = 0,\ i = 1,\ldots,n\}$ and letting

$$\mathbf{X} = \mathbf{I} + \mathbf{P}, \qquad \mathbf{P}\in\mathcal{P} \tag{4.4}$$

[2] Interior-point algorithms are iterative algorithms which terminate once a pre-specified accuracy has been reached. For the SDP in (4.3) the complexity per iteration is $\mathcal{O}(n^3)$ [17, 18], and for a given accuracy the number of iterations required is at most $\mathcal{O}(n^{0.5})$.

Substituting (4.4) into (4.3), the SD relaxation problem is reformulated as:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} Q_{ii} + \sum_{i=1}^{n} \sum_{j=1,\ i\neq j}^{n} P_{ij} Q_{ij} \\ \text{s.t.} \quad & \mathbf{P} \in \mathcal{P} \\ & \mathbf{I} + \mathbf{P} \succeq 0 \end{aligned} \tag{4.5}$$

Obviously, the solution to the SD relaxation problem does not depend on the diagonal elements of $\mathbf{Q}$.

4.2 Approximate Boolean QP Solution via Randomization

We have seen that by replacing the rank-1 constraint of the original problem with a symmetric PSD constraint, SD relaxation leads to an increase in problem dimension. Since the original and relaxed problems have different problem dimensions, some special techniques are required to convert the SD relaxation solution to an approximate Boolean QP solution. A randomization method has been proposed for this conversion process [10, 11]. To gain an intuitive understanding of the randomization, we consider alternative expressions of the Boolean QP and SD relaxation problems. The Boolean QP problem in (4.1) can be expressed as:

$$\max_{\substack{x_i^2=1,\\ i=1,\ldots,n}} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j Q_{ij} \tag{4.6}$$

Define $\mathbf{x}^\star$ to be the solution of (4.6). Notice that $-\mathbf{x}^\star$ is also the solution of (4.6) because both $\mathbf{x}^\star$ and $-\mathbf{x}^\star$ achieve the same objective value. For the SD relaxation problem, let $\mathbf{X} = \mathbf{V}^T\mathbf{V}$ where $\mathbf{V} = [\ \mathbf{v}_1, \ldots, \mathbf{v}_n\ ]$ is any square-root factor (e.g., the Cholesky factor) of $\mathbf{X}$. Substituting $\mathbf{X} = \mathbf{V}^T\mathbf{V}$ into (4.3), we obtain the following equivalent problem:

$$\max_{\substack{\|\mathbf{v}_i\|=1,\\ i=1,\ldots,n}} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{v}_i^T \mathbf{v}_j Q_{ij} \tag{4.7}$$

where $\|.\|$ represents the 2-norm. Define $\hat{\mathbf{X}}$ to be the solution of the SD relaxation problem in (4.3), and $\hat{\mathbf{V}} = [\ \hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_n\ ]$ to be a square-root factor of $\hat{\mathbf{X}}$. Then $\hat{\mathbf{V}}$ is the solution of the equivalent problem in (4.7). Comparing (4.6) with (4.7), an interesting parallel can be observed: the increase in dimensionality in the SD relaxation problem causes the replacement of the scalar product $x_i x_j$ with the inner product $\mathbf{v}_i^T\mathbf{v}_j$. Hence, it appears that one should be able to approximate $x_i^\star$ using $\hat{\mathbf{v}}_i$. In the randomization process described below, $x_i^\star$ is approximated by making a vector-domain threshold decision on $\hat{\mathbf{v}}_i$ with the cutting hyperplane being randomly chosen. Define $\tilde{\mathbf{x}}$ to be the approximate solution generated by the randomization method. The randomization consists of the following steps:

1. Randomly generate a cutting hyperplane which passes through the origin. Such a hyperplane can be represented by the region $\{\mathbf{w} \in \mathbb{R}^n : \mathbf{w}^T\mathbf{u} = 0\}$, where $\mathbf{u}$ is a random vector uniformly distributed on an n-dimensional unit sphere.

2. For $k = 1, \ldots, n$, choose $\tilde{x}_k = -1$ if $\hat{\mathbf{v}}_k$ lies "below" the cutting hyperplane, i.e., $\hat{\mathbf{v}}_k^T\mathbf{u} < 0$; and choose $\tilde{x}_k = 1$ otherwise.

Define $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to be a function where the ith element of $\sigma(\mathbf{x})$ is 1 if $x_i \geq 0$, and -1 otherwise. We notice that the above two steps of the randomization can simply be expressed as:

$$\tilde{\mathbf{x}} = \sigma(\hat{\mathbf{V}}^T\mathbf{u}) \tag{4.8}$$

Usually, to further improve the approximation quality, the randomization is repeated a number of times, and the randomized solution yielding the largest objective function value is chosen as the approximate solution. This procedure is stated in Step 4 of Table 1. Often, this randomization method can achieve an accurate approximation with a modest number of randomizations. For example, in the application of CDMA multiuser detection which will be shown in Section 6, the number of randomizations required to achieve good BER performance is 10 to 20. We also point out that the randomization process (with a modest number of randomizations) is computationally efficient, since its operation count is $\mathcal{O}(n^2 M_{\text{rand}})$, where M_{rand} stands for the number of randomizations. In fact, the complexity of the randomization process (with modest M_{rand}) is almost negligible compared with that of solving the SDP problem.

As we point out in the following property, in some situations it is sufficient to perform the randomization only once.

Property 2 *Let $\mathbf{X} = \mathbf{V}^T\mathbf{V} \in \mathbb{R}^{n\times n}$ with $\mathbf{V} \in \mathbb{R}^{n\times n}$, and let $\mathbf{u}$ be a random vector uniformly distributed on an n-dimensional unit-sphere. For any rank-1 $\mathbf{X}$ such that $\mathbf{X} = \mathbf{x}\mathbf{x}^T$,*

$$\sigma(\mathbf{V}^T\mathbf{u}) = \pm\sigma(\mathbf{x}) \tag{4.9}$$

holds with probability 1.

Proof: Since $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, its square-root factor is determined as

$$\mathbf{V} = [\ x_1\mathbf{z},\ x_2\mathbf{z}, \ldots, x_n\mathbf{z}\ ] \tag{4.10}$$

for some $\|\mathbf{z}\| = 1$. Assuming that $\mathbf{z}^T\mathbf{u} \neq 0$, we have

$$\sigma(\mathbf{V}^T\mathbf{u}) = \sigma(\mathbf{z}^T\mathbf{u})\sigma(\mathbf{x}) \tag{4.11}$$

where $\sigma(\mathbf{z}^T\mathbf{u})$ may either be 1 or -1. To prove that (4.11) holds with probability 1, we consider the probability that $\mathbf{z}^T\mathbf{u} = 0$. According to [10], $\mathbf{u}$ can be obtained by drawing an i.i.d. Gaussian random vector, followed by a normalization; i.e.,

$$\mathbf{u} = \frac{\mathbf{w}}{\|\mathbf{w}\|}, \qquad \mathbf{w} \sim N(0, \mathbf{I}) \tag{4.12}$$

The condition $\mathbf{z}^T\mathbf{u} = 0$ is equivalent to $\mathbf{z}^T\mathbf{w} = 0$. Then, it can be easily shown that the probability $P[\mathbf{z}^T\mathbf{w} = 0]$ tends to zero and thus (4.9) holds with probability 1. $\square$

Applying Property 2 to the SD relaxation algorithm, we see that if $\hat{\mathbf{X}}$ is of rank 1, then, with probability 1, the approximate solution for any number of randomizations is equal to either the sign of the dominant eigenvector of $\hat{\mathbf{X}}$ or its negative version. This special property will be useful in proving an important theorem in the next section.

Finally, we point out that the approximation accuracy of the randomization method has been theoretically analyzed by Nesterov [11]. It was shown that for a sufficient (yet finite) number of randomizations, the approximation accuracy of the randomization method is, at worst, moderate. However, as will be apparent from the simulation results in Section 6, the SD relaxation algorithm often exhibits very good approximation accuracy in practice. Furthermore, the number of randomizations required to achieve good approximation accuracy is often much smaller than that suggested by Nesterov's (worst-case) analysis.

4.3 SD Relaxation ML Detector

To apply the SD relaxation algorithm to the MLD problem, the original MLD problem has to be re-written in the same form as (4.1). Define a scalar $c \in \{-1, 1\}$. Since $c\mathbf{b} \in \{-1, 1\}^K$ for any

$\mathbf{b} \in \{-1,1\}^K$, (2.3) can be rewritten as:

$$\max_{\mathbf{b}\in\{-1,+1\}^K} J(\mathbf{b}) \equiv \max_{\substack{\mathbf{b}\in\{-1,+1\}^K \\ c\in\{-1,+1\}}} J(c\mathbf{b}) \tag{4.13a}$$

$$= \max_{\substack{\mathbf{b}\in\{-1,+1\}^K \\ c\in\{-1,+1\}}} 2c\mathbf{b}^T\mathbf{A}\mathbf{y} - \mathbf{b}^T\mathbf{H}\mathbf{b} \tag{4.13b}$$

$$= \max_{\substack{\mathbf{b}\in\{-1,+1\}^K \\ c\in\{-1,+1\}}} [\ \mathbf{b}^T\ c\ ] \begin{bmatrix} -\mathbf{H} & \mathbf{A}\mathbf{y} \\ (\mathbf{A}\mathbf{y})^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix} \tag{4.13c}$$

Clearly, (4.13c) is equivalent to the Boolean QP problem in (4.1), where $n = K+1$, $\mathbf{x} = [\ \mathbf{b}^T\ c\ ]^T$, and

$$\mathbf{Q} = \begin{bmatrix} -\mathbf{H} & \mathbf{A}\mathbf{y} \\ (\mathbf{A}\mathbf{y})^T & 0 \end{bmatrix} \tag{4.14}$$

Let $(\mathbf{b}^\star, c^\star)$ be the optimal solution of (4.13a). Since $\hat{\mathbf{b}}_{\mathrm{ML}}$ and $c^\star\mathbf{b}^\star$ attain the same maximum objective value in (4.13a),

$$\hat{\mathbf{b}}_{\mathrm{ML}} = c^\star\mathbf{b}^\star \tag{4.15}$$

Hence, we can use the SD relaxation algorithm described in previous subsections to approximate the solution to (4.13c), and then use the relationship in (4.15) to obtain an approximate MLD solution. A summary of the SDR-ML detector is given in Table 1. Finally, we conclude from previous subsections that the complexity of the SDR-ML detector is $\mathcal{O}((K+1)^{3.5}) \simeq \mathcal{O}(K^{3.5})$.

5 Relationship of SDR-ML Detector to Other Multiuser Detectors

In this section, we will consider two relaxation methods called unconstrained relaxation and bound relaxation for the MLD problem. It will be illustrated that some existing multiuser detectors can be viewed as approximate MLD algorithms using these two methods. Then, we will prove that both the unconstrained and bound relaxation methods are further relaxation of the SD relaxation method, and therefore it is expected that the SDR-ML detector should perform better than those existing detectors.

5.1 Unconstrained Relaxation and Bound Relaxation

The unconstrained relaxation (UR) algorithm, which has been briefly described in Section 3, is formulated as

$$\tilde{\mathbf{b}}_{\text{UR}} = \arg \max_{\mathbf{b} \in \mathbb{R}^K} J(\mathbf{b}) \tag{5.1a}$$

$$\hat{\mathbf{b}}_{\text{UR}} = \sigma(\tilde{\mathbf{b}}_{\text{UR}}) \tag{5.1b}$$

where $J(\mathbf{b})$ is given by (2.4), $\hat{\mathbf{b}}_{\text{UR}}$ is the approximate solution using UR, and $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is an element-wise threshold decision function. UR is a loose relaxation since there is no restriction on the possible values of $\tilde{b}_{\text{UR},k}$. An *ad hoc* method that may partially compensate this problem is to add a penalty function to (5.1a):

$$\tilde{\mathbf{b}}_{\text{UR}} = \arg \max_{\mathbf{b} \in \mathbb{R}^K} J(\mathbf{b}) - \gamma \|\mathbf{b}\|^2 \tag{5.2}$$

where $\gamma \|\mathbf{b}\|^2$ is the penalty function with $\gamma \geq 0$ being a constant. The reason for choosing such a penalty function is to implicitly constrain the magnitude of b_k whilst maintaining the least-squares nature of the relaxed problem. Interestingly, the penalized UR problem in (5.2) is still a relaxation of the MLD problem. To illustrate this point, we consider a restriction of (5.2) where the original feasible set is replaced by the alphabet set:

$$\max_{\mathbf{b} \in \{-1,1\}^K} J(\mathbf{b}) - \gamma \|\mathbf{b}\|^2 \tag{5.3}$$

Clearly, Problem (5.3) is an equivalent MLD problem because $\|\mathbf{b}\|^2$ is a constant for any $\mathbf{b} \in \{-1,1\}^K$. Thus, (5.2) is a relaxation of the equivalent MLD problem in (5.3).

As described in Section 3, the major advantage of using UR is the availability of a closed-form solution. Assuming $\mathbf{H} + \gamma \mathbf{I}$ to be a full rank matrix, the solution to (5.2) is given by [16]

$$\tilde{\mathbf{b}}_{\text{UR}} = (\mathbf{H} + \gamma \mathbf{I})^{-1} \mathbf{A} \mathbf{y} \tag{5.4}$$

and it can be shown that

$$\hat{\mathbf{b}}_{\text{UR}} = \sigma((\mathbf{R} + \gamma \mathbf{A}^{-2})^{-1} \mathbf{y}) \tag{5.5}$$

To see how the penalized UR method is related to some of the existing suboptimal detectors, we consider the outputs of three well-known linear detectors; viz. the matched filter detector, the decorrelator and the LMMSE detector, given respectively by

$$\hat{\mathbf{b}}_{\mathrm{MF}} = \sigma(\mathbf{y}) \tag{5.6}$$

$$\hat{\mathbf{b}}_{\mathrm{DC}} = \sigma(\mathbf{R}^{-1}\mathbf{y}) \tag{5.7}$$

$$\hat{\mathbf{b}}_{\mathrm{LMMSE}} = \sigma\Big((\mathbf{R} + \tfrac{\mathcal{N}_o}{2}\mathbf{A}^{-2})^{-1}\mathbf{y}\Big) \tag{5.8}$$

Readers are referred to [1] for the detailed derivations and criteria of the above detectors. Basically, the matched filter detector is a single-user detector which is optimal in the absence of multiuser interference, while the decorrelator and the LMMSE detector use different forms of linear mapping to suppress the multiuser interference.

Comparing (5.5) with (5.6) to (5.8), the equivalence between the UR detector and the linear detectors above can be clearly seen. If the UR problem is not penalized; i.e. $\gamma = 0$, then $\hat{\mathbf{b}}_{\mathrm{UR}} \equiv \hat{\mathbf{b}}_{\mathrm{DC}}$. On the other hand, if γ is chosen to be $\frac{\mathcal{N}_o}{2}$ such that the influence of the penalty function to the UR problem is adjusted according to the noise power, then $\hat{\mathbf{b}}_{\mathrm{UR}} \equiv \hat{\mathbf{b}}_{\mathrm{LMMSE}}$. Finally, if the UR problem is over-penalized such that $\gamma \gg \max E_k$, the approximate solution $\hat{\mathbf{b}}_{\mathrm{UR}}$ approaches $\sigma((\gamma\mathbf{A}^{-2})^{-1}\mathbf{y}) = \sigma(\mathbf{y}) \equiv \hat{\mathbf{b}}_{\mathrm{MF}}$. Thus, each of the linear detectors in (5.6) to (5.8) can be viewed as approximate ML detectors under UR.

Next we consider the bound relaxation (BR) method which has been recently applied to multiuser detection in [14, 13]. Instead of allowing the relaxation solution to lie in $\mathbb{R}^K$, the BR method constrains the relaxation solution to lie within a K-dimensional cube. The BR algorithm is formulated as follows:

$$\tilde{\mathbf{b}}_{\mathrm{BR}} = \arg \max_{\substack{-1 \le b_k \le 1 \\ k=1,\ldots,K}} J(\mathbf{b}) \tag{5.9a}$$

$$\hat{\mathbf{b}}_{\mathrm{BR}} = \sigma(\tilde{\mathbf{b}}_{\mathrm{BR}}) \tag{5.9b}$$

Bound relaxation has two advantages: (i) it is tighter than the unpenalized UR method; and (ii) the relaxed problem has a concave (quadratic) objective function and linear inequality constraints, and hence (5.9a) is a convex optimization problem [19]. There is an existing multiuser detector that is closely related to the BR method. In [9], Nelson and Poor considered the SAGE algorithm as a method of suboptimally solving the MLD problem. It can be seen that one of the modified SAGE detectors, namely the SAGE detector with unit-clipper M-step, actually uses a coordinate ascent method [8] to solve (5.9a), and then takes the sign of the solution to (5.9a) as the approximate MLD solution. (See [15, pp. 89–92] and [13] for the details.)

5.2 Relative Tightness of Various Relaxation Methods

We note that we can represent the UR and BR methods, both with and without a penalty function, by a generalized expression as follows:

$$\begin{aligned} h^\star &= \max_{\substack{-d\le b_k\le d\\ k=1,\ldots,K}} J(\mathbf{b}) - \gamma\|\mathbf{b}\|^2 \\ &= \max_{\substack{-d\le b_k\le d\\ k=1,\ldots,K}} 2\mathbf{b}^T\mathbf{A}\mathbf{y} - \mathbf{b}^T(\mathbf{H}+\gamma\mathbf{I})\mathbf{b} \end{aligned} \tag{5.10}$$

where $d \ge 1$ and $\gamma \ge 0$ are constants, and $h^\star$ denotes the maximum objective value achieved by Problem (5.10). We call (5.10) the generalized bound relaxation (GBR) problem. It is observed that if $\mathbf{H}+\gamma\mathbf{I}$ is positive definite, then the objective function in (5.10) is strictly concave and thus the solution to (5.10) is unique [19]. (Note that if $\mathbf{R}$ is of full rank, or if $\gamma > 0$, then $\mathbf{H}+\gamma\mathbf{I}$ is positive definite.) Let $\tilde{\mathbf{b}}_{\text{GBR}}$ be the solution of (5.10) and let $\hat{\mathbf{b}}_{\text{GBR}} = \sigma(\tilde{\mathbf{b}}_{\text{GBR}})$ be the approximate MLD solution of the GBR problem. By following the same procedure as described in Section 4.3, Problem (5.10) can be re-written as

$$\begin{aligned} h^\star = &\max_{\substack{x_k^2\le d^2,\ k=1,\ldots,K\\ x_{K+1}^2=1}} \quad \mathbf{x}^T\tilde{\mathbf{Q}}\mathbf{x} \\ = \max \quad & \text{Trace}(\mathbf{X}\tilde{\mathbf{Q}}) \\ \text{s.t.} \quad & \mathbf{X} = \mathbf{x}\mathbf{x}^T \\ & X_{ii} \le d^2, \qquad i = 1,\ldots,K \\ & X_{K+1,K+1} = 1 \end{aligned} \tag{5.11}$$

where $\mathbf{x} = [\ \mathbf{b}^T\ \ c\ ]^T$ with $c \in \{-1, 1\}$, and

$$\tilde{\mathbf{Q}} = \begin{bmatrix} -(\mathbf{H}+\gamma\mathbf{I}) & \mathbf{A}\mathbf{y} \\ (\mathbf{A}\mathbf{y})^T & 0 \end{bmatrix} \tag{5.12}$$

If the rank-1 constraint in (5.11) is replaced by a symmetric PSD constraint, we obtain an SD relaxation of the GBR problem

$$\begin{aligned} l^\star = \max \quad & \text{Trace}(\mathbf{X}\tilde{\mathbf{Q}}) \\ \text{s.t.} \quad & \mathbf{X} \succeq \mathbf{0} \\ & X_{ii} \le d^2, \qquad i = 1,\ldots,K \\ & X_{K+1,K+1} = 1 \end{aligned} \tag{5.13}$$

where $l^\star$ is the maximum objective value achieved by (5.13). Let $\hat{\hat{\mathbf{X}}}$ be the solution of (5.13) and let $\hat{\hat{\mathbf{b}}}_{\text{GBR}}$ be an approximate MLD solution obtained from applying the randomization procedure in Section 4.2 to $\hat{\hat{\mathbf{X}}}$. We have the following equivalence theorem:

Theorem 1 *The GBR problem given by (5.10) and its SD relaxed version given by (5.13) are equivalent in that they achieve the same maximum objective value; i.e.,*

$$h^\star = l^\star. \tag{5.14}$$

Furthermore, if $\mathbf{H} + \gamma\mathbf{I}$ is positive definite, then, with probability 1,

$$\hat{\hat{\mathbf{b}}}_{\text{GBR}} = \hat{\mathbf{b}}_{\text{GBR}} \tag{5.15}$$

for any number of randomizations.

Proof: We first prove (5.14) by showing $h^\star \le l^\star$ and $h^\star \ge l^\star$. The former is obvious since (5.13) is a relaxation of (5.10). To show the latter, we consider a reformulation of (5.13). Let

$$\mathbf{X} = \begin{bmatrix} \mathbf{Z} & \boldsymbol{\zeta} \\ \boldsymbol{\zeta}^T & 1 \end{bmatrix} \tag{5.16}$$

where $\mathbf{Z} \in \mathbb{R}^{K\times K}$ is a symmetric matrix, and $\boldsymbol{\zeta} \in \mathbb{R}^K$. We note that $\mathbf{X} \succeq \mathbf{0}$ is equivalent to $\mathbf{Z} - \boldsymbol{\zeta}\boldsymbol{\zeta}^T \succeq \mathbf{0}$ due to the Schur complement [20]. By substituting (5.16) into (5.13), Problem (5.13) can be reformulated as:

$$\begin{aligned} l^\star = \max\ & 2\boldsymbol{\zeta}^T\mathbf{A}\mathbf{y} - \text{Trace}(\mathbf{Z}(\mathbf{H}+\gamma\mathbf{I})) \\ \text{s.t. } & \mathbf{Z} - \boldsymbol{\zeta}\boldsymbol{\zeta}^T \succeq \mathbf{0} \\ & Z_{ii} \le d^2, \quad i = 1,\dots,K \end{aligned} \tag{5.17}$$

Since $\mathbf{H} + \gamma\mathbf{I} \succeq \mathbf{0}$,

$$\text{Trace}((\mathbf{Z} - \boldsymbol{\zeta}\boldsymbol{\zeta}^T)(\mathbf{H}+\gamma\mathbf{I})) \ge 0 \tag{5.18}$$

for any $\mathbf{Z} - \boldsymbol{\zeta}\boldsymbol{\zeta}^T \succeq \mathbf{0}$. It follows from (5.18) that

$$\text{Trace}(\mathbf{Z}(\mathbf{H}+\gamma\mathbf{I})) \ge \boldsymbol{\zeta}^T(\mathbf{H}+\gamma\mathbf{I})\boldsymbol{\zeta} \tag{5.19}$$

for any feasible $(\mathbf{Z}, \boldsymbol{\zeta})$. Furthermore, since the diagonal elements of a PSD matrix are non-negative,

$$\zeta_i{}^2 \leq Z_{ii} \leq d^2, \quad i = 1, \ldots, K \tag{5.20}$$

for any feasible $(\mathbf{Z}, \boldsymbol{\zeta})$. Now, define $(\mathbf{Z}^\star, \boldsymbol{\zeta}^\star)$ to be the solution of (5.17). Applying (5.19) and (5.20) to (5.17), we get

$$l^\star \leq 2(\boldsymbol{\zeta}^\star)^T \mathbf{A}\mathbf{y} - (\boldsymbol{\zeta}^\star)^T(\mathbf{H} + \gamma\mathbf{I})\boldsymbol{\zeta}^\star \tag{5.21a}$$

$$\leq \max_{\zeta_i{}^2 \leq d^2 \ \forall \ i} 2\boldsymbol{\zeta}^T \mathbf{A}\mathbf{y} - \boldsymbol{\zeta}^T(\mathbf{H} + \gamma\mathbf{I})\boldsymbol{\zeta} = h^\star \tag{5.21b}$$

Thus, (5.14) follows.

Next we prove the equivalence in (5.15), using the assumption that the matrix $\mathbf{H} + \gamma\mathbf{I}$ is positive definite. Since $h^\star = l^\star$, the equality sign holds in (5.21a) and (5.21b). Comparing (5.17) with (5.21a), we see that $(\mathbf{Z}^\star, \boldsymbol{\zeta}^\star)$ must satisfy the following condition:

$$\mathrm{Trace}(\mathbf{Z}^\star(\mathbf{H} + \gamma\mathbf{I})) - (\boldsymbol{\zeta}^\star)^T(\mathbf{H} + \gamma\mathbf{I})\boldsymbol{\zeta}^\star = 0 \tag{5.22a}$$

which is equivalent to

$$\mathrm{Trace}((\mathbf{Z}^\star - \boldsymbol{\zeta}^\star(\boldsymbol{\zeta}^\star)^T)(\mathbf{H} + \gamma\mathbf{I})) = 0 \tag{5.22b}$$

Using standard properties of positive definite and semi-definite matrices [21, p. 318], it can be shown that (5.22b) holds if and only if

$$(\mathbf{Z}^\star - \boldsymbol{\zeta}^\star(\boldsymbol{\zeta}^\star)^T)(\mathbf{H} + \gamma\mathbf{I}) = 0 \tag{5.23}$$

Hence, for a positive definite (and thus invertible) $\mathbf{H} + \gamma\mathbf{I}$, we must have

$$\mathbf{Z}^\star = \boldsymbol{\zeta}^\star(\boldsymbol{\zeta}^\star)^T \tag{5.24}$$

Subsequently, the solution to (5.13), denoted by $\hat{\hat{\mathbf{X}}}$, is of the form

$$\hat{\hat{\mathbf{X}}} = \mathbf{q}\mathbf{q}^\mathbf{T}, \quad \mathbf{q} = [\ (\boldsymbol{\zeta}^\star)^\mathbf{T}\ \mathbf{1}\ ]^\mathbf{T} \tag{5.25}$$

Comparing the two equivalent problems (5.10) and (5.17), we see that $\boldsymbol{\zeta}^\star = \tilde{\mathbf{b}}_{\mathrm{GBR}}$. We also notice that $\boldsymbol{\zeta}^\star$ is unique because $\tilde{\mathbf{b}}_{\mathrm{GBR}}$ is unique for a positive definite $\mathbf{H} + \gamma\mathbf{I}$. Finally, by applying

Property 2 to (5.25), it can be shown that for any number of randomizations, (5.15) holds with probability 1. □

Theorem 1 provides the important implication that the SD relaxation algorithm for the GBR problem is equivalent to the GBR algorithm itself in the sense that they have the same tightness, and that they produce the same approximate solution under the assumption of $\mathbf{H}+\gamma\mathbf{I}$ being positive definite. More importantly, due to Property 1 (in Section 4.1), the SD relaxation problem in (4.3) can be re-expressed as

$$\begin{aligned} \max \quad & \mathrm{Trace}(\mathbf{X}\tilde{\mathbf{Q}}) \\ \text{s.t.} \quad & \mathbf{X} \succeq \mathbf{0} \\ & X_{ii} = 1, \qquad i = 1, \ldots, n \end{aligned} \tag{5.26}$$

where $\tilde{\mathbf{Q}}$ is given by (5.12). Obviously, the equivalent GBR problem in (5.13) is a relaxation of (5.26) where K of the equality constraints in (5.26) are replaced by inequality constraints. Hence, SD relaxation is tighter than GBR and therefore it is expected that the SDR-ML detector should yield a better approximation than any GBR-based detectors. The superior performance of the SDR-ML detector is demonstrated in the following section.

6 Simulation Results

We now demonstrate the performance of the SDR-ML detector in a number of synchronous CDMA scenarios.

Example 1 The purpose of this example is to compare the BER performance of the ML detector and that of the SDR-ML detector. Since the ML detector is computationally prohibitive to implement for large K, we are restricted to the case of a small number of users. In this example, we consider a 4-user system with spreading factor equal to 7. The corresponding signal correlation matrix is given by

$$\mathbf{R} = \frac{1}{7}\begin{bmatrix} 7 & -1 & 3 & 3 \\ -1 & 7 & 3 & -1 \\ 3 & 3 & 7 & -1 \\ 3 & -1 & -1 & 7 \end{bmatrix} \tag{6.1}$$

We are interested in a near-far simulation scenario in which the BER performance of a particular user is evaluated under various interfering user signal energies. The simulation setting is similar to that in [9] and is given as follows: User 3 is chosen to be the desired user with SNR fixed at $2E_3/\mathcal{N}_o = 11$dB, while users 1, 2 and 4 are treated as interferers with $2E_1/\mathcal{N}_o = 2E_2/\mathcal{N}_o = 2E_4/\mathcal{N}_o$.

(Similar results are obtained when one of the other users is chosen as the desired user.) The number of trials for the simulation is set to be 1,000,000. The near-far performance of the true ML detector (implemented via exhaustive search) and the SDR-ML detector is plotted in Fig. 1(a), where the symbol M_{rand} represents the number of randomizations. For reference, the no-interference lower bound is plotted in the same figure. (The no-interference bound is the performance of user 3 in the absence of users 1, 2 and 4, and hence is independent of the abscissa.) It is seen that the performance of the SDR-ML detector improves significantly with M_{rand}, and that its BER for $M_{\mathrm{rand}} = 10$ is almost the same as that of the true ML detector. We also compare the BER performance of the SDR-ML detector with that of various suboptimal multiuser detectors in Fig. 1(b), using the same scenario as in Fig. 1(a). Recall from Section 5 that the matched filter detector, the decorrelator, and the LMMSE detector can be regarded as unconstrained-relaxation detectors, while the SAGE detector with unit-clipper M-step can be viewed as a bound-relaxation detector. (It should also be pointed out that the SAGE detector with hard-decision M-step is structurally equivalent to a serial-update interference-canceling detector [9].) It is seen that the SDR-ML detector provides better BER performance than these alternative suboptimal detectors, which supports our viewpoint in Section 5 that SD relaxation should perform better than unconstrained relaxation and bound relaxation. □

Example 2 In the second example, we are interested in average BER performance of a number of multiuser detectors when all users have the same signal strength. Two sets of spreading codes, namely the length-31 Gold codes with $K = 16$, and the length-15 Kasami codes (large set) with $K = 15$, are chosen for the test. Measures of the cross-correlations of the two spreading code sets are tabulated in Table 2. It is observed that the length-15 Kasami codes exhibit a much stronger cross-correlation than the length-31 Gold codes. Figs. 2 and 3 show the average BER performance of the various multiuser detectors for the two code sets. For reference, we have also plotted the no-interference lower bound and Verdú's theoretical upper bound [1] on the BER performance of the true ML detector. We first observe that the SDR-ML detector provides good BER performance for $M_{\mathrm{rand}} \geq 10$. Moreover, it is seen that the BER performance of the SDR-ML detector is better than that of the other multiuser detectors, particularly in the case of length-15 Kasami codes, where the performance of other detectors is considerably degraded by the strong signal cross-correlations. In the case of length-31 Gold codes, the BER of the SDR-ML detector ($M_{\mathrm{rand}} \geq 15$) lies well below Verdú's upper bound [1] on the BER of the true ML detector. In the case of length-15 Kasami codes, the BER of the SDR-ML detector (M_{rand}=20) is below Verdú's upper bound for $2E_k/\mathcal{N}_o < 10$dB, and slightly above Verdú's upper bound at $2E_k/\mathcal{N}_o = 11$dB. Since Verdú's upper bound is tight for sufficiently high SNRs [1], we infer that the performance difference between the SDR-ML detector and the true ML detector should be small. (Recall that the computational cost of empirically evaluating the BER performance of the exact ML detector is prohibitive in this scenario.) □

Example 3 In the third example, we use the length-31 Gold codes and the length-15 Kasami codes in the previous example to test the near-far performance of the various detectors. The simulation setting is as follows: Users 1 to 8 are the interferers with $2E_1/\mathcal{N}_o = 2E_2/\mathcal{N}_o = \ldots = 2E_8/\mathcal{N}_o$. Users 9 to K are the desired users with SNRs fixed at $2E_9/\mathcal{N}_o = 2E_{10}/\mathcal{N}_o = \ldots = 2E_K/\mathcal{N}_o = 11$dB. The number of trials for the simulation is 1,000,000. Due to space limitations, we illustrate the average near-far performance of the desired users in Fig. 4, instead of showing the near-far performance plots for each user. Again, the SDR-ML detector is seen to provide better average BER performance than the other suboptimal detectors. Though not shown here, the near-far performance of the SDR-ML detector for each user was also observed to be better than that of the other detectors. □

7 Conclusion and Discussion

In this paper, we have applied the SD relaxation method to approximately solve the NP-hard MLD problem with a polynomial-time complexity of $\mathcal{O}(K^{3.5})$. Simulation results have shown that the SDR-ML detector achieves BER performance close to that of the true ML detector, even when the signal cross-correlations are strong or the near-far effect is significant. Moreover, we have shown that some existing detectors such as the decorrelator, the LMMSE detector, and a particular form of the modified SAGE detector, can be considered as degenerate forms of the SDR-ML detector. The resulting expectation that the SDR-ML detector should perform better than those detectors was confirmed by simulations which showed that the SDR-ML detector often provides substantially improved performance. Since those existing detectors require $\mathcal{O}(K^3)$ operations [c.f., (5.7), (5.8)], the SDR-ML detector offers an attractive tradeoff between BER performance and computational cost.

The work presented here leads to a few interesting future directions. First, this work has focussed on the application of SD relaxation to synchronous CDMA systems. It will be interesting to see how the SDR-ML detection technique may be applied to other scenarios, such as CDMA over frequency selective fading channels and asynchronous CDMA systems. Second, we have implemented the SDR-ML detector using a standard algorithm [17, 18]. It remains to be seen whether fast implementations of this algorithm can be developed for SDR-ML detection applications.

Figures and Tables

Table 1: A summary of the SDR-ML detector.

Step 1. Set $n = K + 1$, and

$$\mathbf{Q} = \begin{bmatrix} -\mathbf{H} & \mathbf{Ay} \\ (\mathbf{Ay})^T & 0 \end{bmatrix}$$

Step 2. Solve the semi-definite program

$$\hat{\mathbf{X}} = \arg \max_{\substack{\mathbf{X} \succeq 0 \\ X_{ii}=1 \ \forall \ i}} \ \mathrm{Trace}(\mathbf{XQ})$$

Step 3. Factorize $\hat{\mathbf{X}} = \hat{\mathbf{V}}^T \hat{\mathbf{V}}$.

Step 4. Randomization: Denote the number of randomizations by M_{rand}.

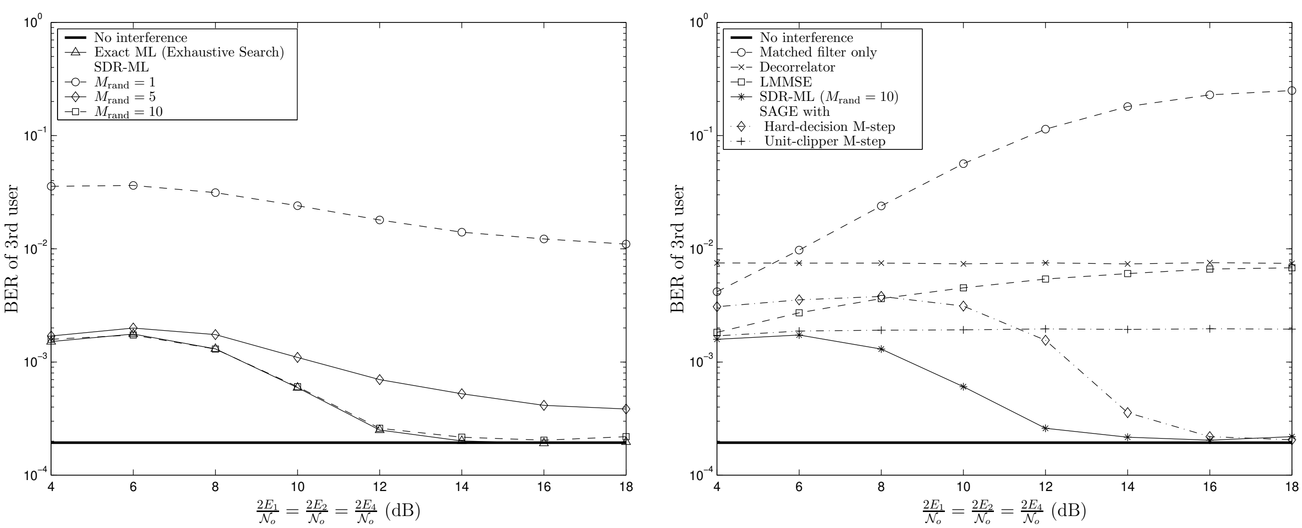
for i*=1,2,...,*M_{rand}

Randomly generate a vector $\mathbf{u}_i$ uniformly distributed on an n-dimensional unit sphere. Compute $\tilde{\mathbf{x}}_i = \sigma(\hat{\mathbf{V}}^T \mathbf{u}_i)$.

end;

Choose $\hat{\mathbf{x}} = \tilde{\mathbf{x}}_j$ as the approximate solution of $\mathbf{x}^\star$, where $j = \arg \max_{i=1,\ldots,M_{\mathrm{rand}}} \tilde{\mathbf{x}}_i^T \mathbf{Q} \tilde{\mathbf{x}}_i$.

Step 5. Take $\hat{\mathbf{b}}_{\text{SDR}} = \hat{x}_{K+1}[\ \hat{x}_1, \ldots, \hat{x}_K\ ]^T$ as the approximate MLD solution.

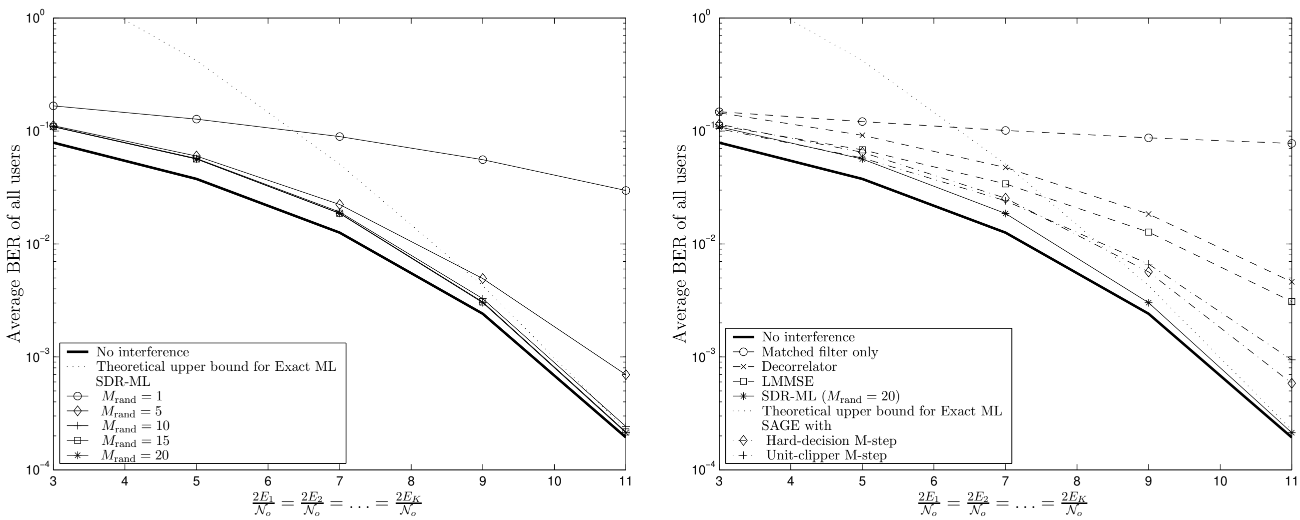


(a) (b)

Figure 1: Near-far performance of (a) the SDR-ML detector and the true ML detector; and (b) various multiuser detectors. The number of users is 4, and the user of interest is 3 with $2E_3/\mathcal{N}_o$=11dB. Note that for the SAGE detectors, decorrelator initialization is employed and the number of stages is 5.

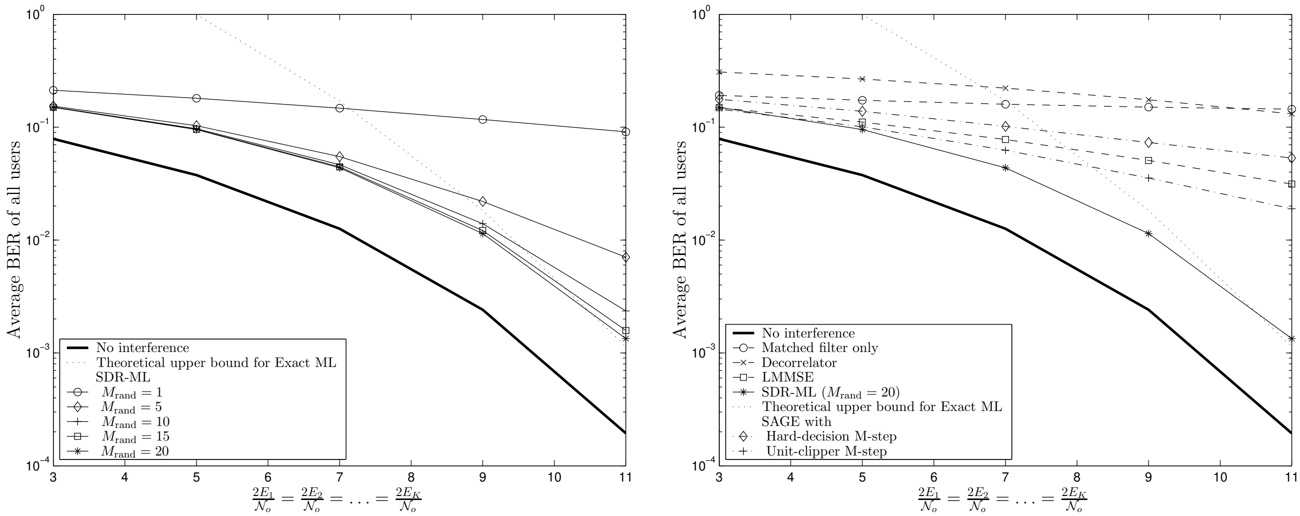
Table 2: Maximum-absolute cross-correlation values and sum-squared cross-correlation values of two spreading codes. Gold-31 and Kas-15 respectively stand for the length-31 Gold codes with $K = 16$, and the length-15 Kasami codes (large set) with $K = 15$.

User	$\max_{j \neq i} \lvert R_{ij} \rvert$		$\sum_{j \neq i} R_{ij}^2$	
Index (i)	Gold-31	Kas-15	Gold-31	Kas-15
1	0.226	0.6	0.365	1.38
2	0.29	0.333	0.515	0.276
3	0.226	0.6	0.265	1.34
4	0.29	0.467	0.415	0.916
5	0.226	0.467	0.415	0.56
6	0.29	0.467	0.382	0.524
7	0.29	0.6	0.515	0.667
8	0.29	0.6	0.349	1.06
9	0.29	0.467	0.582	0.56
10	0.29	0.333	0.399	0.347
11	0.29	0.6	0.682	0.916
12	0.29	0.6	0.332	1.59
13	0.29	0.467	0.232	0.56
14	0.29	0.6	0.332	0.88
15	0.29	0.467	0.365	0.88
16	0.29	-	0.465	-



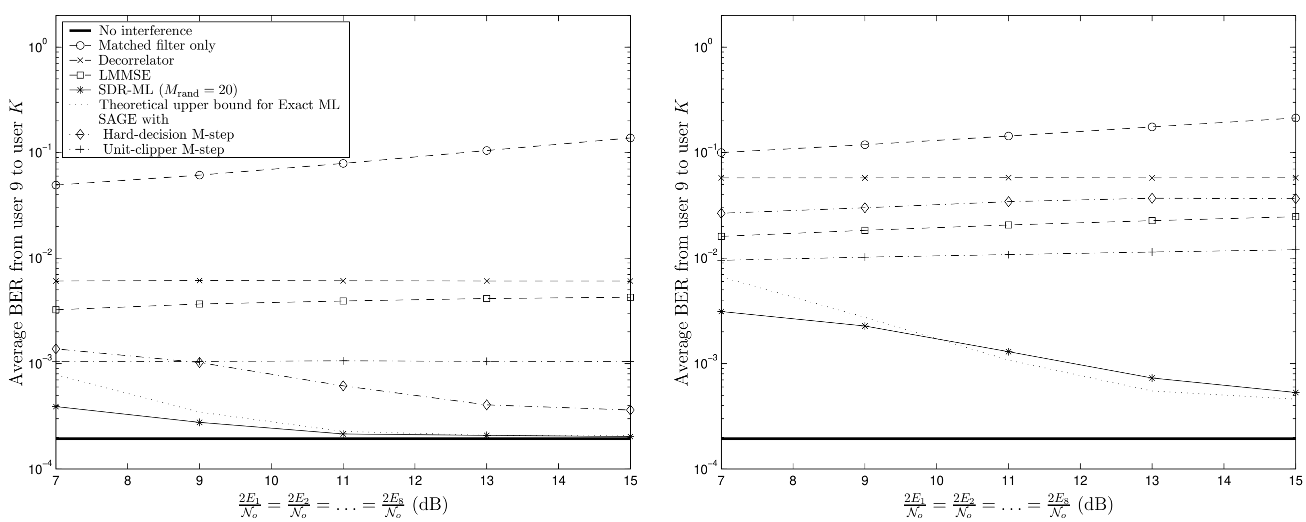
(a) (b)

Figure 2: BER performance in a 16-user synchronous CDMA system using length-31 Gold codes. (a) Average BER of the SDR-ML detector with different number of randomizations. (b) Average BERs of various multiuser detectors. Note that for the SAGE detectors, decorrelator initialization is employed and the number of stages is 5.



(a) (b)

Figure 3: BER performance in a 15-user synchronous CDMA system using length-15 Kasami codes. (a) Average BER of the SDR-ML detector with different number of randomizations. (b) Average BERs of various multiuser detectors. Note that for the SAGE detectors, decorrelator initialization is employed and the number of stages is 5.



(a) (b)

Figure 4: Near-far performance plots illustrating the average BERs of the desired users versus the SNRs of the interfering users. (a) 16-user synchronous CDMA system using length-31 Gold codes. (b) 15-user synchronous CDMA system using length-15 Kasami codes. Note that for the SAGE detectors, decorrelator initialization is employed and the number of stages is 5.

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