Technical Report for "Antenna Subset Selection Optimization for Large-scale MISO Constant Envelope Precoding"

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This technical report provides the proof of Proposition 1 in [1] which claims that the following optimization problem is NP-hard.

$$\begin{aligned} \min_{\boldsymbol{a}\in\mathbb{R}^N} \quad & \boldsymbol{a}^T\mathbf{1} && \text{(1a)}\\ \text{s.t.} \quad & \tfrac{\sqrt{2}|s|_{\max}}{\eta\sqrt{P_{\rm PA}}}\rho + 2\tfrac{|s|_{\max}}{\eta}\boldsymbol{\epsilon}^T\boldsymbol{a} \le \boldsymbol{g}^T\boldsymbol{a} && \text{(1b)}\\ & \tfrac{2|s|_{\max}}{|s|_{\min}+|s|_{\max}}\|\boldsymbol{g}\odot\boldsymbol{a}\|_\infty \le \boldsymbol{g}^T\boldsymbol{a} && \text{(1c)}\\ & \boldsymbol{a}\in\{0,1\}^N, && \text{(1d)} \end{aligned}$$

where $\boldsymbol{g}\in\mathbb{R}_+^N$, $\boldsymbol{\epsilon}\in\mathbb{R}_+^N$ and $\eta > \mathbb{R}_+$. For convenience, let us use the following notations:

$$\begin{aligned} s &= \tfrac{\sqrt{2}|s|_{\max}}{\eta\sqrt{P_{\rm PA}}}\rho\\ t &= \tfrac{2|s|_{\max}}{|s|_{\min}+|s|_{\max}} - 1\\ \boldsymbol{p} &= \boldsymbol{g} - 2\tfrac{|s|_{\max}}{\eta}\boldsymbol{\epsilon}\\ \mathcal{I} &= \{i \mid a_i = 1\}. \end{aligned}$$

Then, problem (1) can be written as

$$\min_{\mathcal{I}} \quad |\mathcal{I}| \tag{2a}$$

$$\text{s.t.} \quad s \le \sum_{i\in\mathcal{I}} p_i \tag{2b}$$

$$tg_{\bar{i}} \le \sum_{i\in\mathcal{I},\ i\neq\bar{i}} g_i, \quad \bar{i} = \arg\max_{i\in\mathcal{I}} g_i \tag{2c}$$

$$\mathcal{I} \subset \{0,\dots,N\}. \tag{2d}$$

Our strategy is to show that the decision version of (2) is NP-complete by reducing the knapsack problem to it. The decision version of (2) is as follows: Given a positive k, determine if there exists an index set $\mathcal{I} \subset \{1,\dots,N\}$ such that

$$\begin{cases} |\mathcal{I}| \le k \\ s \le \sum_{i\in\mathcal{I}} p_i \\ tg_{\bar{i}} \le \sum_{i\in\mathcal{I}, i\neq\bar{i}} g_i, \quad \bar{i} = \arg\max_{i\in\mathcal{I}} g_i. \end{cases} \tag{3}$$

The knapsack problem is as follows: Given a number $C \in \mathbb{Z}_+$, determine if there exists an index set $\mathcal{I}' \subset \{1,\dots,N'\}$ such that

$$\begin{cases} \sum_{i\in\mathcal{I}'} c_i \ge C \\ \sum_{i\in\mathcal{I}'} w_i \le W, \end{cases} \tag{4}$$

where $\boldsymbol{c} \in \mathbb{Z}_+^{N'}$ and $\boldsymbol{w} \in \mathbb{Z}_+^{N'}$.

Given an instance $\mathcal{J}' = (\boldsymbol{c}, \boldsymbol{w}, C, W)$ of the knapsack problem, construct an instance $\mathcal{J} = (\boldsymbol{p}, \boldsymbol{g}, s, t, k)$ of problem (3) by

$$\begin{aligned}
&g_i = 2c_i, && p_i = -w_i, && \text{for all } i = 1,\dots,N' \\
&g_{N'+1} = 1, && p_{N'+1} = 1 + W + \textstyle\sum_{i=1}^{N'} w_i \\
&g_{N'+2} = 2\max_{i=1,\dots,N'} c_i, && p_{N'+2} = 1 + W + 2\textstyle\sum_{i=1}^{N'} w_i \\
&g_{N'+3} = 2C + 2\max_{i=1,\dots,N'} c_i, && p_{N'+3} = 2(1 + W + 2\textstyle\sum_{i=1}^{N'} w_i) \\
&s = 4 + 3W + 7\textstyle\sum_{i=1}^{N'} w_i \\
&t = 1 \\
&N = N' + 3, && k = N' + 3.
\end{aligned}$$

Obviously this construction can be computed in polynomial time. We proceed to show that $\mathcal{J}'$ is a yes instance if and only if $\mathcal{J}$ is a yes instance, or equivalently the following two set of conditions are equivalent:

- Condition 1: there is an index set $\mathcal{I} \subset \{1, \dots, N\}$ such that

$$\begin{cases} |\mathcal{I}| \le k & \text{(5a)} \\ s \le \sum_{i \in \mathcal{I}} p_i & \text{(5b)} \\ t g_{\bar{i}} \le \sum_{i \in \mathcal{I}, i \neq \bar{i}} g_i, \quad \bar{i} = \arg\max_{i \in \mathcal{I}} g_i. & \text{(5c)} \end{cases}$$

- Condition 2: there is an index set $\mathcal{I}' \subset \{1, \dots, N'\}$ such that

$$\begin{cases} \sum_{i \in \mathcal{I}'} c_i \ge C & \text{(6a)} \\ \sum_{i \in \mathcal{I}'} w_i \le W. & \text{(6b)} \end{cases}$$

We first show that condition 2 implies condition 1. Let $\mathcal{I}'$ be an index set that satisfies condition 2. Let us verify that $\mathcal{I} = \mathcal{I}' \bigcup \{N'+1, N'+2, N'+3\}$ satisfies condition 1. Clearly the (5a) is satisfied.

For (5b), consider the following inequality

$$\begin{aligned} &\sum_{i \in \mathcal{I}} p_i \\ =& p_{N'+1} + p_{N'+2} + p_{N'+3} + \sum_{i \in \mathcal{I}'} p_i \\ =& 4 + 4W + 7\sum_{i=1}^{N'} w_i - \sum_{i \in \mathcal{I}'} w_i \\ \ge& 4 + 3W + 7\sum_{i=1}^{N'} w_i \\ =& s, \end{aligned}$$

where the inequality is due to (6b).

For (5c), note that $\bar{i} = N'+3$ by construction. Then, we have

$$\begin{aligned} &\sum_{i \in \mathcal{I}, i \neq N'+3} g_i \\ =& g_{N'+1} + g_{N'+2} + \sum_{i \in \mathcal{I}'} g_i \\ =& 1 + 2\max_{i=1,\dots,N'} c_i + 2\sum_{i \in \mathcal{I}'} c_i \\ \ge& 1 + 2\max_{i=1,\dots,N'} c_i + 2C \\ \ge& t g_{N'+3}, \end{aligned}$$

where the first inequality is due to (6a).

We then show that condition 1 implies condition 2. Let $\mathcal{I}$ be an index set that satisfies condition 1. Let us show that $\{N'+1, N'+2, N'+3\}$ belongs to $\mathcal{I}$. Suppose not, then

$$\begin{aligned}
&\sum_{i\in\mathcal{I}} p_i \\
\leq & p_{N'+2} + p_{N'+3} + \sum_{i\in\mathcal{I}\backslash\{N'+1,N'+2,N'+3\}} p_i \\
=&3\left(1 + W + 2\sum_{i=1}^{N'} w_i\right) - \sum_{i\in\mathcal{I}\backslash\{N'+1,N'+2,N'+3\}} w_i \\
<&s,
\end{aligned}$$

where the first inequality is due to $p_{N'+3} \geq p_{N'+2} \geq p_{N'+1}$. This result contradicts (5b). Hence, it follows have that $\{N'+1, N'+2, N'+3\}$ belongs to $\mathcal{I}'$.

We then verify that $\mathcal{I}' = \mathcal{I}\backslash\{N'+1, N'+2, N'+3\}$ satisfies condition 2. Note that we have $\bar{i} = N'+3$ by construction. For (6a), consider

$$\begin{aligned}
& \sum_{i\in\mathcal{I}, i\neq\bar{i}} g_i \geq t g_{\bar{i}} \\
\iff \quad & g_{N'+1} + g_{N'+2} + \sum_{i\in\mathcal{I}'} g_i \geq g_{N'+3} \\
\iff \quad & 2\sum_{i\in\mathcal{I}'} c_i \geq 2C - 1 \\
\iff \quad & \sum_{i\in\mathcal{I}'} c_i \geq C.
\end{aligned}$$

where the last step is due to the fact that C and all c_i are integers. For (6b), consider

$$\begin{aligned}
& \sum_{i\in\mathcal{I}} p_i \geq s \\
\iff \quad & p_{N'+1} + p_{N'+2} + p_{N'+3} + \sum_{i\in\mathcal{I}'} p_i \geq s \\
\iff \quad & -\sum_{i\in\mathcal{I}'} w_i \geq -W.
\end{aligned}$$

REFERENCES

[1] J. Pan and W.-K. Ma, "Antenna subset selection optimization for large-scale MISO constant envelope precoding," in *Proc. IEEE Int. Conf. Acoustic, Speech, Signal Process. (ICASSP)*, 2014, to appear.