CHEAP SEMIDEFINITE RELAXATION MIMO DETECTION USING ROW-BY-ROW BLOCK COORDINATE DESCENT

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ABSTRACT

This paper considers the problem of low complexity implementation of high-performance semidefinite relaxation (SDR) MIMO detection methods. Currently, most SDR MIMO detectors are implemented using interior-point methods. Although such implementations have worst-case polynomial complexity (approximately cubic in the problem size), they can be quite computationally costly in practice. Here we depart from the interior-point method framework and investigate the use of other low per-iteration-complexity techniques for SDR MIMO detection. Specifically, we employ the row-by-row (RBR) method, which is a particular version of block coordinate descent, to solve the semidefinite programs that arise in the SDR MIMO context with an emphasis on the QPSK scenario. In each iteration of the RBR method, only matrix-vector multiplications are needed, and hence it can be implemented in a very efficient manner. Our simulation results show that the RBR method can indeed offer a significant speedup in runtime, while providing bit error rate performance on par with the interior-point methods.

Index Terms— semidefinite relaxation, MIMO detection, block coordinate descent

1. INTRODUCTION

In MIMO detection, semidefinite relaxation (SDR) [1] has been recognized as an efficient, high-performance approach. SDR is an approximate maximum-likelihood (ML) approach based on semidefinite programs (SDPs), which can be solved in time polynomial in the problem size. Numerical evidence has suggested that SDR can yield very competitive symbol error rate performance, especially in the BPSK and QPSK scenarios. A general coverage of the SDR technique, including applications other than MIMO detection, can be found in [2]; another overview focusing on MIMO detection is provided in the introduction of [3]. Readers are also referred to the references therein regarding the current theoretical and practical advances of SDR MIMO detection.

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In SDR MIMO detection, an aspect that has caught recent interest is fast implementation. A standard way of implementing the SDR detector is to use an interior-point method (IPM) to solve the SDP. Indeed, IPMs are efficient and reliable SDP solvers in the sense that they can produce numerically accurate solutions in polynomial time; e.g., $\mathcal{O}(n^{3.5})$ in the BPSK and QPSK scenarios, where n is the problem size. However, we should note at this point that high solution precision is arguably not a critical factor in practice, since the SDP solution will typically have to go through some kind of symbol rounding process (see [2] for details). Moreover, the sophistication of the interior-point technique, which is the reason for good solution fidelity, means that IPMs are generally not low complexity options for implementing the SDR detector. In view of this, some attempts have been made to reduce the complexity required to solve the SDR problem, such as the early termination methods in [4, 5] and the custom-built IPM in [6].

In this paper, we depart from the IPM framework and investigate the use of other low per-iteration complexity techniques for SDR MIMO detection. Specifically, we consider the row-by-row (RBR) method [7], which is recently developed in the optimization community for solving SDPs. Our contribution lies in bringing this new tool to SDR MIMO detection in the QPSK scenario (also applicable to BPSK), with an emphasis on demonstrating its potential in practice. A distinguishing feature of the RBR method in the context of SDR MIMO detection is that the arithmetic operations in each iteration are inexpensive, requiring only simple matrix-vector multiplications whose total complexity is $\mathcal{O}(n^3)$. Our simulations show that the SDR detector implemented by the RBR method can yield a bit error rate (BER) performance very close to that implemented by an IPM, yet the former provides a complexity saving of about 10 times relative to the latter. To encourage the readers to try out the RBR method and get a sense of its performance, we provide the source codes of the SDR-RBR detector at `http://www.ee.cuhk.edu.hk/~wkma/mimo`.

Most of the notations in this paper are standard. We use $\mathbb{S}^n$ to denote the set of $n \times n$ symmetric matrices, $\| \cdot \|$ to denote the 2-norm of a vector and $\mathbf{X} \succeq \mathbf{0}$ to indicate that $\mathbf{X}$ is symmetric positive semidefinite (PSD).

2. PROBLEM STATEMENT

2.1. Problem Formulation

This work follows a standard, but widely encountered MIMO system formulation. We consider an MIMO signal model

$$\mathbf{y}_C = \mathbf{H}_C \mathbf{s}_C + \mathbf{v}_C. \tag{1}$$

Here, $\mathbf{y}_C \in \mathbb{C}^M$ is the receive vector, $\mathbf{s}_C \in \mathcal{S}^N$ is the transmitted symbol vector, $\mathbf{H}_C \in \mathbb{C}^{M\times N}$ is the MIMO channel, $\mathbf{v}_C$ is a noise vector assumed to be white circular complex Gaussian, M is the receive dimension, N is the number of transmitted symbols and $\mathcal{S}$ is a symbol constellation set. It has been known in the literature [8] that model (1) covers a wide variety of detection problems in multiuser and multi-antenna communications. We are interested in the ML detection for the QPSK scenario; i.e., $\mathcal{S} = \{\pm 1 \pm j\}$. Let

$$\mathbf{y} = \begin{bmatrix} \Re\{\mathbf{y}_C\} \\ \Im\{\mathbf{y}_C\} \end{bmatrix}, \mathbf{s} = \begin{bmatrix} \Re\{\mathbf{s}_C\} \\ \Im\{\mathbf{s}_C\} \end{bmatrix}, \mathbf{H} = \begin{bmatrix} \Re\{\mathbf{H}_C\} & -\Im\{\mathbf{H}_C\} \\ \Im\{\mathbf{H}_C\} & \Re\{\mathbf{H}_C\} \end{bmatrix}.$$

The ML detector for model (1) is given by

$$\min_{\mathbf{s}\in\{\pm1\}^{2N}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2. \tag{2}$$

The ML problem (2) is computationally difficult, although it is optimal in yielding the minimum error probability of detecting $\mathbf{s}$. There are many different approaches to handling the ML problem, such as the sphere decoding approach [8], the lattice reduction aided approach [9] and the SDR approach.

2.2. Semidefinite Relaxation

This work concentrates on the SDR approach, which has been empirically proven to have near-optimal ML performance in the QPSK scenario [1, 2]. In that approach, the ML problem (2) is first rewritten as a homogeneous quadratic program:

$$\min_{\mathbf{x}\in\{\pm1\}^n} \mathbf{x}^T\mathbf{C}\mathbf{x}, \tag{3}$$

where $n = 2N+1$,

$$\mathbf{x} = \begin{bmatrix} \mathbf{s} \\ t \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} \mathbf{H}^T\mathbf{H} & -\mathbf{H}^T\mathbf{y} \\ -\mathbf{y}^T\mathbf{H} & \|\mathbf{y}\|^2 \end{bmatrix}$$

and t is an augmented variable for homogenizing (2). Notice that (2) and (3) are equivalent under the correspondence $\mathbf{s} = x_n\mathbf{x}_{1:n-1}$. Then, we consider the following SDR of (3):

$$\begin{aligned} \min_{\mathbf{X}\in\mathbb{S}^n} \quad & \mathrm{Tr}(\mathbf{C}\mathbf{X}) \\ \text{s.t.} \quad & \mathbf{X} \succeq \mathbf{0}, \quad X_{ii} = 1,\ i = 1,\ldots,n. \end{aligned} \tag{4}$$

Problem (4) is a relaxation of Problem (3) because

$$\mathbf{X} = \mathbf{x}\mathbf{x}^T, \mathbf{x} \in \{\pm1\}^n \Longrightarrow \mathbf{X} \succeq \mathbf{0}, X_{ii} = 1,\ i = 1,\ldots,n,$$

but the converse is not necessarily true. The SDR problem (4) is convex and tractable; specifically, it is an SDP. After the SDR problem is solved, we can extract from the SDR solution an approximate solution to (3) using, e.g., the Gaussian randomization method. For the approximation procedure, readers are referred to the literature [1, 2] for details.

In SDR MIMO detection, the bulk of the computational cost lies in solving the SDP (4). At present, interior-point methods (IPMs) constitute a predominant class of solvers for SDP. The interior-point approach is efficient and reliable in the sense that i) given a solution accuracy $\epsilon > 0$, an IPM is guaranteed to terminate in $\mathcal{O}(n^{1/2}\log(1/\epsilon))$ iterations; and ii) the computational complexity per iteration is $\mathcal{O}(n^3)$[1]. Despite these appealing characteristics, IPMs are not considered low complexity methods. Indeed, a close inspection of an IPM—such as the primal-dual IPM in [10]—reveals that each iteration requires computing an explicit $n\times n$ matrix inverse, solving an $n\times n$ linear system of equations and performing some line searches.

In the next section we will consider an alternative SDR solver that has low complexity per iteration.

3. THE ROW-BY-ROW METHOD

The row-by-row (RBR) method [7] is a recently proposed technique for solving SDPs. It is in principle a block coordinate descent method, and it can exploit the special diagonal constraint structures of the SDR problem (4) to reduce the complexity of its iterate updates. To describe the method, consider the following barrier SDR problem:

$$\begin{aligned} \min_{\mathbf{X}\in\mathbb{S}^n} \quad & \mathrm{Tr}(\mathbf{C}\mathbf{X}) - \sigma\log\det(\mathbf{X}) \\ \text{s.t.} \quad & X_{ii} = 1, \quad i = 1,\ldots,n, \end{aligned} \tag{5}$$

where $\sigma > 0$ is called the barrier parameter. As is common in IPMs, the log determinant function in (5) is to ensure that $\mathbf{X}$ remains in the interior of the set of PSD matrices. The parameter σ controls the approximation accuracy of the barrier SDR problem (5) relative to the original SDR problem (4). Specifically, let $f^\star$ and f_σ denote the optimal objective values of Problems (4) and (5), respectively. It is known that [11]

$$|f^\star - f_\sigma| \le n\sigma. \tag{6}$$

The RBR method solves the barrier SDR problem (5) by applying cyclic optimization over the rows (or columns) of $\mathbf{X}$—at each step, we optimize (5) with respect to (w.r.t.) only one row of $\mathbf{X}$, while holding other variables of $\mathbf{X}$ fixed. For ease of exposition of the RBR update, let us first consider the partial optimization of (5) w.r.t. the first row. By symmetry, the matrices $\mathbf{X}$ and $\mathbf{C}$ can be partitioned as

$$\mathbf{X} = \begin{bmatrix} X_{11} & \boldsymbol{\xi}_1^T \\ \boldsymbol{\xi}_1 & \bar{\mathbf{X}}_{11} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} C_{11} & \mathbf{c}_1^T \\ \mathbf{c}_1 & \bar{\mathbf{C}}_{11} \end{bmatrix}.$$

The partial optimization of (5) w.r.t. the first row of $\mathbf{X}$ can be written as

$$\min_{\boldsymbol{\xi}_1\in\mathbb{R}^{n-1}} \quad 2\mathbf{c}_1^T\boldsymbol{\xi}_1 - \sigma\log(1 - \boldsymbol{\xi}_1^T\bar{\mathbf{X}}_{11}^\dagger\boldsymbol{\xi}_1), \tag{7}$$

[1]To obtain such a complexity per iteration, one needs to exploit the SDR problem structures; see, e.g., the primal-dual IPM in [10].

where the Schur complement $\det(\mathbf{X}) = \det(\bar{\mathbf{X}}_{11})\det(X_{11} - \boldsymbol{\xi}_1^T \bar{\mathbf{X}}_{11}^\dagger \boldsymbol{\xi}_1)$ and $X_{11} = 1$ have been used to obtain (7). Problem (7) is an unconstrained convex problem, and its optimal solution can be found using the first order optimality condition. Let $f_1(\boldsymbol{\xi}_1)$ denote the objective function of (7). The gradient of $f_1(\boldsymbol{\xi}_1)$ is

$$\nabla f_1(\boldsymbol{\xi}_1) = 2\mathbf{c}_1 - \frac{2\sigma}{1 - \boldsymbol{\xi}_1^T \bar{\mathbf{X}}_{11}^\dagger \boldsymbol{\xi}_1} \bar{\mathbf{X}}_{11}^\dagger \boldsymbol{\xi}_1.$$

By solving $\nabla f_1(\boldsymbol{\xi}_1) = \mathbf{0}$ for $\boldsymbol{\xi}_1$, we see that Problem (7) has a closed form solution. Specifically, let $\gamma = \mathbf{c}_1^T \bar{\mathbf{X}}_{11} \mathbf{c}_1$. The optimal solution to Problem (7) is

$$\boldsymbol{\xi}_1^\star = \begin{cases} -\frac{1}{2\gamma}\left(\sqrt{\sigma^2 + 4\gamma} - \sigma\right)\bar{\mathbf{X}}_{11}\mathbf{c}_1 & \text{if } \gamma > 0, \\ \mathbf{0} & \text{if } \gamma = 0. \end{cases} \quad (8)$$

The partial minimization of the barrier SDR (5) w.r.t. the other rows yields exactly the same result as above (with differences only in notations). We summarize the RBR method in Algorithm 1. We use $\bar{\mathbf{X}}_{ii} \in \mathbb{S}^{n-1}$ to denote the submatrix of $\mathbf{X}$ with the ith column and ith row removed. The vector $\boldsymbol{\xi}_i \in \mathbb{R}^{n-1}$ (resp. $\mathbf{c}_i \in \mathbb{R}^{n-1}$) is obtained by taking the ith column of $\mathbf{X}$ (resp. $\mathbf{C}$) and then removing the ith row. From Algorithm 1, one can see that each row update requires one matrix-vector multiplication and one vector-vector multiplication, which costs $\mathcal{O}(n^2)$ in computations. Since the row updates are performed n times for each iteration, we conclude that the total complexity per iteration is $\mathcal{O}(n^3)$. In fact, the per-iteration complexity is low and comparable to that of a linear detector.

Algorithm 1: Row-by-row method for Problem (5)

Input: $\mathbf{X}^{(0)} \succ \mathbf{0}$ – initialization,
$\sigma > 0$ – barrier parameter
set $k = 1$, $f^{(0)} = \mathrm{Tr}(\mathbf{C}\mathbf{X}^{(0)})$ and $\mathbf{X}^{(1)} = \mathbf{X}^{(0)}$
repeat
  for *i = 1,2,...,n* **do**
    set $\mathbf{z} = \bar{\mathbf{X}}_{ii}^{(k)} \mathbf{c}_i$ and $\gamma = \mathbf{z}^T \mathbf{c}_i$
    if $\gamma > 0$ **then**
      set $\boldsymbol{\xi}_i^{(k)} = -\frac{1}{2\gamma}\left(\sqrt{\sigma^2 + 4\gamma} - \sigma\right)\mathbf{z}$
    else
      set $\boldsymbol{\xi}_i^{(k)} = \mathbf{0}$
  compute $f^{(k)} = \mathrm{Tr}(\mathbf{C}\mathbf{X}^{(k)})$, set $\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)}$
  and $k = k + 1$
until *some termination criterion is satisfied*;
Output: $\mathbf{X}^{(k)}$ – solution to (5)

The iterates produced by Algorithm 1 are known to converge to the optimal solution to the barrier SDR (5). Specifically, we have

$$\lim_{k \to \infty} \mathbf{X}^{(k)} = \mathbf{X}_\sigma,$$

where $\mathbf{X}_\sigma$ is the optimal solution to (5). Note that the result is concerned with asymptotic convergence, but not the convergence rate. It has been known that in a general application context, block coordinate descent can be slow in terms of convergence speed. Curiously, we noticed by simulations that for the MIMO detection application here, the number of iterations required to obtain a good BER performance is rather modest, say, within 10 iterations. This will be demonstrated in detail in the next section.

4. SIMULATION RESULTS

In this section, we evaluate the BER and complexity performance of the proposed RBR-based SDR detector by Monte Carlo simulations. In particular, we compare its performance with that of the IPM-based SDR detector.

The simulation settings are as follows. The channel $\mathbf{H}_C$ is randomly generated following an i.i.d. complex Gaussian distribution with zero mean and unit variance. The barrier parameter of the SDR-RBR detector is set to be

$$\sigma = 10^{-2}/n.$$

With this setting, the target solution accuracy of the RBR method is 10^{-2}; cf. (6). The initialization is $\mathbf{X}^{(0)} = \mathbf{I}$. We choose a standard stopping criterion for the RBR method:

$$\left| \frac{f^{(k+1)} - f^{(k)}}{f^{(k)}} \right| \leq \delta, \quad (9)$$

where $\delta > 0$ is given. We fix $\delta = 10^{-2}$. Moreover, we implement the SDR-IPM detector by employing the primal-dual IPM in [10]. The IPM solution accuracy is also set at 10^{-2}. The solution rounding in SDR-RBR and SDR-IPM is done by the Gaussian randomization method [1, 2], with $\max\{10, 2n\}$ randomizations. To provide some benchmark, the sphere decoder, the lattice reduction aided (LRA) MMSE-DF detector, and the zero-forcing (ZF) detector were also tested. The detectors are written mainly in C, with minor operations relying on MATLAB. The simulations were done on a 3.0GHz dual-core desktop computer with 2GB of memory.

Fig. 1(a) shows the BERs of the various detectors under an MIMO system size of $(M, N) = (40, 40)$. Note that such a problem size may happen in multiuser systems, and that sphere decoding is computationally too slow to run in this case. As can be seen, the BERs of the SDR-RBR and SDR-IPM detectors are very close. We tested other problem size settings and found the same result. Moreover, the two SDR detectors outperforms the other detectors significantly.

Since the SDR-RBR detector provides almost the same BERs as the SDR-IPM detector, we are interested in the complexity of the former. To study this, we first change the RBR stopping criterion to a fixed number of iterations, and then evaluate how the BERs are affected by the number of iterations. The results are shown in Fig. 1(b). We see that the SDR-RBR detector attains outstanding performance when the number of iterations is around 10.

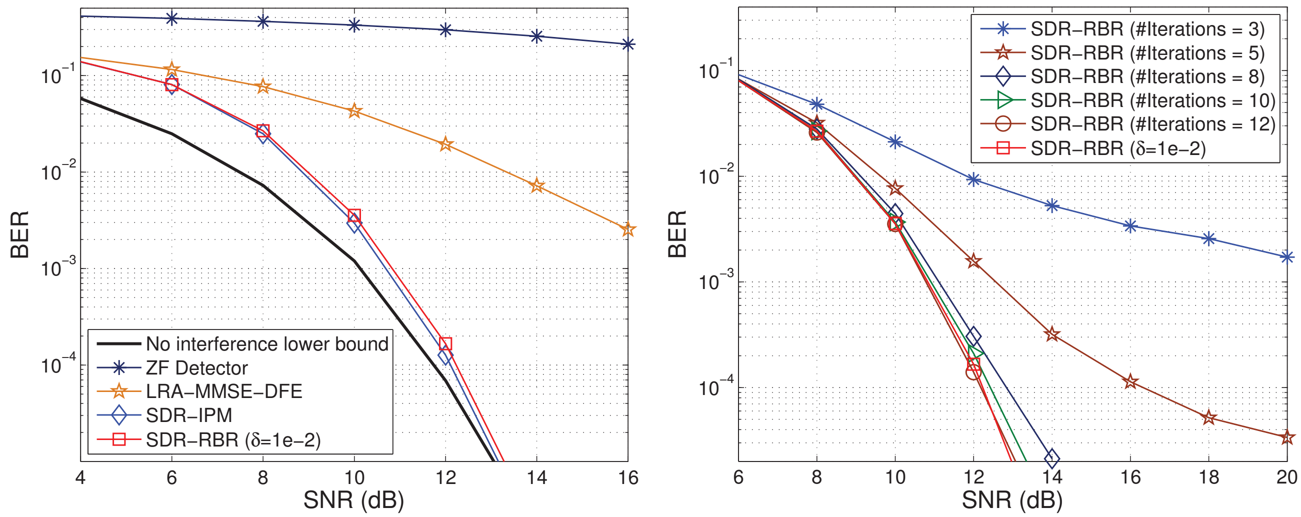

Fig. 1. BERs in a QPSK 40×40 system. (a) Comparison of various detectors; (b) SDR-RBR for various number of iterations.

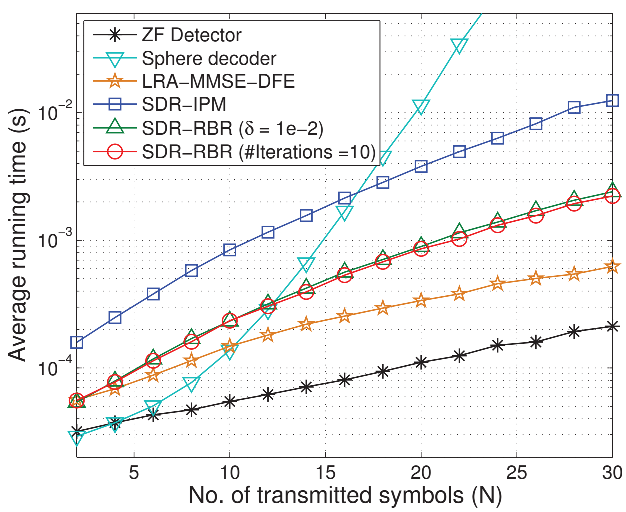

Fig. 2. Computational complexities of the various detectors.

In the last simulation we examine the computational complexities of the various detectors with respect to the problem size. We fix SNR= 12dB and $M = N$. The results are shown in Fig. 2. It can be seen that the SDR-RBR detector, either with the stopping rule (9) or with a fixed number of iterations of 10, is much faster than the SDR-IPM detector; the difference is about 10 times.

5. CONCLUSION

In this paper, we have demonstrated the potential of the RBR method for low complexity implementations of the SDR MIMO detector. In particular, we have shown by simulations that the RBR method can lead to a tenfold runtime saving when compared to the conventional IPM implementation. Such empirically attractive results not only prove the feasibility of the RBR approach, but also paves the way for real-world implementations of the SDR detector.

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