# ENGG 5781 Matrix Analysis and Computations Lecture 9: Kronecker Product

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# **Lecture 9: Kronecker Product**

- Kronecker product and properties
- vectorization
- Kronecker sum

# **Motivating Problem: Matrix Equations**

 $\bullet$  Problem: given A, B, find an X such that

$$\mathbf{A}\mathbf{X} = \mathbf{B}.$$
 (\*)

- an easy problem; if A has full column rank and (\*) has a solution, the solution is merely  $X = A^{\dagger}B$ .
- Question: but how about matrix equations like
  - $-\mathbf{AX} + \mathbf{XB} = \mathbf{C},$
  - $\mathbf{A}_1 \mathbf{X} \mathbf{B}_1 + \mathbf{A}_2 \mathbf{X} \mathbf{B}_2 = \mathbf{C}$ ,
  - AX + YB = C, X, Y both being unknown?
- such matrix equations can be tackled via matrix tools arising from the Kronecker product

## **Kronecker Product**

The Kronecker product of  ${\bf A}$  and  ${\bf B}$  is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & & a_{2n}\mathbf{B} \\ \vdots & & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}$$

• Example: let  $\mathbf{a} \in \mathbb{R}^m$ ,  $\mathbf{b} \in \mathbb{R}^m$ . By definition,

$$\mathbf{a} \otimes \mathbf{b} = egin{bmatrix} a_1 \mathbf{b} \ a_2 \mathbf{b} \ dots \ a_m \mathbf{b} \end{bmatrix}$$

Note that, since

$$\mathbf{b}\mathbf{a}^T = [a_1\mathbf{b}, a_2\mathbf{b}, \dots, a_m\mathbf{b}],$$

 $\mathbf{a} \otimes \mathbf{b}$  is a column-by-column concatenation of the outer product  $\mathbf{b}\mathbf{a}^T$ .

# **Properties**

Elementary properties:

- 1.  $\mathbf{A} \otimes (\alpha \mathbf{B}) = (\alpha \mathbf{A}) \otimes \mathbf{B}$ .
- 2.  $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$ ,  $\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$  (distributive)
- 3.  $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$  (associativity).
- 4.  $\mathbf{0}_{mn} = \mathbf{0}_m \otimes \mathbf{0}_n$ ,  $\mathbf{I}_{mn} = \mathbf{I}_m \otimes \mathbf{I}_n$ ;  $\mathbf{0}_n$  and  $\mathbf{I}_n$  are  $n \times n$  zero and identity matrices.
- 5.  $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$ ,  $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$ .
- 6. there exist permutation matrices  $\mathbf{U}_1$  and  $\mathbf{U}_2$  such that

$$\mathbf{U}_1(\mathbf{A}\otimes\mathbf{B})\mathbf{U}_2=\mathbf{B}\otimes\mathbf{A}.$$

Note: Kronecker product is not commutative; i.e.,  $\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$  in general. Property 6 above is a weak version of commutativity.

# **More Properties**

Property 9.1 (mixed product rule).

 $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=(\mathbf{A}\mathbf{C})\otimes(\mathbf{B}\mathbf{D}),$ 

for A, B, C, D of appropriate matrix dimensions.

Some properties from Property 9.1:

1. if  $\mathbf{A} \in \mathbb{R}^{m \times m}$  and  $\mathbf{B} \in \mathbb{R}^{n \times n}$  are nonsingular, then

$$(\mathbf{A}\otimes\mathbf{B})^{-1}=\mathbf{A}^{-1}\otimes\mathbf{B}^{-1}$$

- proof:  $(\mathbf{A}^{-1} \otimes \mathbf{B}^{-1})(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{A}^{-1}\mathbf{A}) \otimes (\mathbf{B}^{-1}\mathbf{B}) = \mathbf{I}_m \otimes \mathbf{I}_n = \mathbf{I}_{mn}.$ 

2. if  $\mathbf{Q}_1$ ,  $\mathbf{Q}_2$  are semi-orthogonal, then  $\mathbf{Q}_1 \otimes \mathbf{Q}_2$  is semi-orthogonal.

- proof:  $(\mathbf{Q}_1 \otimes \mathbf{Q}_2)^T (\mathbf{Q}_1 \otimes \mathbf{Q}_2) = (\mathbf{Q}_1^T \otimes \mathbf{Q}_2^T) (\mathbf{Q}_1 \otimes \mathbf{Q}_2) = (\mathbf{Q}_1^T \mathbf{Q}_1) \otimes (\mathbf{Q}_1^T \mathbf{Q}_1) = \mathbf{I}.$ 

# **Example: Hadamard Matrix**

Consider an  $2\times 2$  orthogonal matrix

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1\\ 1 & -1 \end{bmatrix}.$$

From  $\mathbf{H}_2$ , construct a  $4 \times 4$  matrix

and inductively,  $\mathbf{H}_n = \mathbf{H}_{n/2} \otimes \mathbf{H}_{n/2}$  for any n that is a power of 2.

- is  $\mathbf{H}_4$  orthogonal? Yes, because  $\mathbf{H}_4\mathbf{H}_4^T = (\mathbf{H}_2 \otimes \mathbf{H}_2)(\mathbf{H}_2^T \otimes \mathbf{H}_2^T) = (\mathbf{H}_2\mathbf{H}_2^T \otimes \mathbf{H}_2^T) = \mathbf{I}.$
- for the same reason, any  $\mathbf{H}_n$  is orthogonal

#### **Kronecker Product and Eigenvalues**

**Theorem 9.1.** Let  $\mathbf{A} \in \mathbb{R}^{m \times m}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times n}$ . Let  $\{\lambda_i, \mathbf{x}_i\}_{i=1}^m$  be the set of m eigen-pairs of  $\mathbf{A}$ , and let  $\{\mu_i, \mathbf{y}_i\}_{i=1}^n$  be the set of n eigen-pairs of  $\mathbf{B}$ . The set of mn eigen-pairs of  $\mathbf{A} \otimes \mathbf{B}$  is given by

$$\{\lambda_i \mu_j, \mathbf{x}_i \otimes \mathbf{y}_j\}_{i=1,\dots,m, j=1,\dots,n}$$

Properties arising from Theorem 9.1 (for square A, B):

- 1.  $\det(\mathbf{A} \otimes \mathbf{B}) = [\det(\mathbf{A})]^n [\det(\mathbf{B})]^m$ .
- 2.  $tr(\mathbf{A} \otimes \mathbf{B}) = tr(\mathbf{A})tr(\mathbf{B})$ .
- 3. if A and B are (symmetric) PSD, then  $A \otimes B$  is PSD.

## Vectorization

The vectorization of  $\mathbf{A} \in \mathbb{R}^{m \times n}$  is defined as

$$\operatorname{vec}(\mathbf{A}) = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix},$$

i.e., we stack the columns of a matrix to form a column vector.

Property 9.2.  $vec(AXB) = (B^T \otimes A)vec(X)$ .

Special cases of Property 9.2:

 $\operatorname{vec}(\mathbf{A}\mathbf{X}) = (\mathbf{I} \otimes \mathbf{A})\operatorname{vec}(\mathbf{X})$  $\operatorname{vec}(\mathbf{X}\mathbf{A}) = (\mathbf{A}^T \otimes \mathbf{I})\operatorname{vec}(\mathbf{X})$ 

# **Proof Sketch of Property 9.2**

• write

$$\mathbf{X} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \mathbf{e}_i \mathbf{e}_j^T$$

• by letting  $\mathbf{a}_i$  be the *i*th column of  $\mathbf{A}$  and  $\mathbf{b}_j$  the *j*th row of  $\mathbf{B}$ ,

$$\operatorname{vec}(\mathbf{AXB}) = \operatorname{vec}\left(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \mathbf{A} \mathbf{e}_{i} \mathbf{e}_{j}^{T} \mathbf{B}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \operatorname{vec}(\mathbf{a}_{i} \mathbf{b}_{j}^{T}).$$

• by noting

$$\operatorname{vec}(\mathbf{a}_{i}\mathbf{b}_{j}^{T}) = \operatorname{vec}([\mathbf{a}_{i}b_{j1},\ldots,\mathbf{a}_{i}b_{j,q}]) = \begin{bmatrix} b_{j1}\mathbf{a}_{i} \\ \vdots \\ b_{j,q}\mathbf{a}_{i} \end{bmatrix} = \mathbf{b}_{j} \otimes \mathbf{a}_{i}$$

we get  $\operatorname{vec}(\mathbf{AXB}) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \mathbf{b}_j \otimes \mathbf{a}_i = (\mathbf{B}^T \otimes \mathbf{A}) \operatorname{vec}(\mathbf{X}).$ 

## **Kronecker Sum**

• Problem: given  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{m \times m}$ ,  $\mathbf{C} \in \mathbb{R}^{n \times m}$ , solve

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C} \tag{*}$$

with respect to  $\mathbf{X} \in \mathbb{R}^{m \times n}$ .

• the above problem is a linear system. By vectorizing (\*), we get

$$(\mathbf{I}_m \otimes \mathbf{A}) \operatorname{vec}(\mathbf{X}) + (\mathbf{B}^T \otimes \mathbf{I}_n) \operatorname{vec}(\mathbf{X}) = \operatorname{vec}(\mathbf{C})$$

• the Kronecker sum of  $\mathbf{A} \in \mathbb{R}^{n imes n}$  and  $\mathbf{B} \in \mathbb{R}^{m imes m}$  is

$$\mathbf{A} \oplus \mathbf{B} = (\mathbf{I}_m \otimes \mathbf{A}) + (\mathbf{B} \otimes \mathbf{I}_n).$$

– if a unique solution to (\*) is desired, we wish to know conditions under which  ${f A}\oplus {f B}$  is nonsingular

## **Kronecker Sum**

**Theorem 9.2.** Let  $\{\lambda_i, \mathbf{x}_i\}_{i=1}^n$  be the set of n eigen-pairs of  $\mathbf{A}$ , and let  $\{\mu_i, \mathbf{y}_i\}_{i=1}^m$  be the set of m eigen-pairs of  $\mathbf{B}$ . The set of mn eigen-pairs of  $\mathbf{A} \oplus \mathbf{B}$  is given by

$$\{\lambda_i + \mu_j, \mathbf{y}_j \otimes \mathbf{x}_i\}_{i=1,\dots,n, j=1,\dots,m}$$

**Theorem 9.3.** The matrix equations

AX + XB = C

has a unique solution for every given  ${\bf C}$  if and only if

$$\lambda_i 
eq -\mu_j,$$
 for all  $i, j$ ,

where  $\{\lambda_i\}_{i=1}^n$  and  $\{\mu_i\}_{i=1}^m$  are the set of eigenvalues of A and B, resp.

• idea behind Theorem 9.3: if  $\lambda_i = -\mu_j$  for some i, j, then from Theorem 9.2 there exists a zero eigenvalue for  $\mathbf{A} \oplus \mathbf{B}$ .

## **Kronecker Sum**

• Consider

$$\mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} = \mathbf{C},$$

which is called the Lyapunov equations.

• from Theorem 9.3, the Lyapunov equations admit a unique solution if

$$\lambda_i \neq -\lambda_j$$
, for all  $i, j$ .

• if A is PD such that  $\lambda_i > 0$  for all *i*, the Lyapunov equations always have a unique solution.