ENGG5781 Matrix Analysis and Computations
Lecture 9: Kronecker Product

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Lecture 9: Kronecker Product

- Kronecker product and properties
- vectorization
- Kronecker sum

Motivating Problem: Matrix Equations

- Problem: given $\mathbf{A}$, $\mathbf{B}$, find an $\mathbf{X}$ such that

$$\mathbf{AX} = \mathbf{B}. \tag{$*$}$$

  - an easy problem; if $\mathbf{A}$ has full column rank and $(*)$ has a solution, the solution is merely $\mathbf{X} = \mathbf{A}^\dagger \mathbf{B}$.

- Question: but how about matrix equations like
  - $\mathbf{AX} + \mathbf{XB} = \mathbf{C}$,
  - $\mathbf{A}_1\mathbf{XB}_1 + \mathbf{A}_2\mathbf{XB}_2 = \mathbf{C}$,
  - $\mathbf{AX} + \mathbf{YB} = \mathbf{C}$, $\mathbf{X}, \mathbf{Y}$ both being unknown?

- such matrix equations can be tackled via matrix tools arising from the Kronecker product

Kronecker Product

The Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & & a_{2n}\mathbf{B} \\ \vdots & & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}.$$

- Example: let $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^m$. By definition,

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1\mathbf{b} \\ a_2\mathbf{b} \\ \vdots \\ a_m\mathbf{b} \end{bmatrix}$$

  Note that, since

$$\mathbf{b}\mathbf{a}^T = [\ a_1\mathbf{b}, a_2\mathbf{b}, \dots, a_m\mathbf{b}\ ],$$

  $\mathbf{a} \otimes \mathbf{b}$ is a column-by-column concatenation of the outer product $\mathbf{b}\mathbf{a}^T$.

Properties

Elementary properties:

1. $\mathbf{A} \otimes (\alpha \mathbf{B}) = (\alpha \mathbf{A}) \otimes \mathbf{B}$.
2. $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$, $\quad \mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$ (distributive)
3. $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$ (associativity).
4. $\mathbf{0}_{mn} = \mathbf{0}_m \otimes \mathbf{0}_n$, $\mathbf{I}_{mn} = \mathbf{I}_m \otimes \mathbf{I}_n$; $\mathbf{0}_n$ and $\mathbf{I}_n$ are $n \times n$ zero and identity matrices.
5. $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$, $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$.
6. there exist permutation matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ such that

$$\mathbf{U}_1 (\mathbf{A} \otimes \mathbf{B}) \mathbf{U}_2 = \mathbf{B} \otimes \mathbf{A}.$$

Note: Kronecker product is not commutative; i.e., $\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$ in general. Property 6 above is a weak version of commutativity.

More Properties

Property 9.1 (mixed product rule)**.**

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}),$$

for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ of appropriate matrix dimensions.

Some properties from Property 9.1:

1. if $\mathbf{A} \in \mathbb{R}^{m \times m}$ and $\mathbf{B} \in \mathbb{R}^{n \times n}$ are nonsingular, then

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$$

   – proof: $(\mathbf{A}^{-1} \otimes \mathbf{B}^{-1})(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{A}^{-1}\mathbf{A}) \otimes (\mathbf{B}^{-1}\mathbf{B}) = \mathbf{I}_m \otimes \mathbf{I}_n = \mathbf{I}_{mn}$.

2. if $\mathbf{Q}_1$, $\mathbf{Q}_2$ are semi-orthogonal, then $\mathbf{Q}_1 \otimes \mathbf{Q}_2$ is semi-orthogonal.

   – proof: $(\mathbf{Q}_1 \otimes \mathbf{Q}_2)^T(\mathbf{Q}_1 \otimes \mathbf{Q}_2) = (\mathbf{Q}_1^T \otimes \mathbf{Q}_2^T)(\mathbf{Q}_1 \otimes \mathbf{Q}_2) = (\mathbf{Q}_1^T\mathbf{Q}_1) \otimes (\mathbf{Q}_1^T\mathbf{Q}_1) = \mathbf{I}$.

Example: Hadamard Matrix

Consider an 2×2 orthogonal matrix

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

From $\mathbf{H}_2$, construct a 4×4 matrix

$$\mathbf{H}_4 = \mathbf{H}_2 \otimes \mathbf{H}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

and inductively, $\mathbf{H}_n = \mathbf{H}_{n/2} \otimes \mathbf{H}_{n/2}$ for any n that is a power of 2.

- is $\mathbf{H}_4$ orthogonal? Yes, because $\mathbf{H}_4\mathbf{H}_4^T = (\mathbf{H}_2 \otimes \mathbf{H}_2)(\mathbf{H}_2^T \otimes \mathbf{H}_2^T) = (\mathbf{H}_2\mathbf{H}_2^T \otimes \mathbf{H}_2\mathbf{H}_2^T) = \mathbf{I}$.
- for the same reason, any $\mathbf{H}_n$ is orthogonal

Kronecker Product and Eigenvalues

Theorem 9.1. Let $\mathbf{A} \in \mathbb{R}^{m \times m}$, $\mathbf{B} \in \mathbb{R}^{n \times n}$. Let $\{\lambda_i, \mathbf{x}_i\}_{i=1}^m$ be the set of m eigen-pairs of $\mathbf{A}$, and let $\{\mu_i, \mathbf{y}_i\}_{i=1}^n$ be the set of n eigen-pairs of $\mathbf{B}$. The set of mn eigen-pairs of $\mathbf{A} \otimes \mathbf{B}$ is given by

$$\{\lambda_i \mu_j, \mathbf{x}_i \otimes \mathbf{y}_j\}_{i=1,\ldots,m,\ j=1,\ldots,n}$$

Properties arising from Theorem 9.1 (for square $\mathbf{A}, \mathbf{B}$):

1. $\det(\mathbf{A} \otimes \mathbf{B}) = [\det(\mathbf{A})]^n [\det(\mathbf{B})]^m$.
2. $\mathrm{tr}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})$.
3. if $\mathbf{A}$ and $\mathbf{B}$ are (symmetric) PSD, then $\mathbf{A} \otimes \mathbf{B}$ is PSD.

Vectorization

The vectorization of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is defined as

$$\text{vec}(\mathbf{A}) = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix},$$

i.e., we stack the columns of a matrix to form a column vector.

Property 9.2. $\text{vec}(\mathbf{AXB}) = (\mathbf{B}^T \otimes \mathbf{A})\text{vec}(\mathbf{X})$.

Special cases of Property 9.2:

$$\text{vec}(\mathbf{AX}) = (\mathbf{I} \otimes \mathbf{A})\text{vec}(\mathbf{X})$$
$$\text{vec}(\mathbf{XA}) = (\mathbf{A}^T \otimes \mathbf{I})\text{vec}(\mathbf{X})$$

Proof Sketch of Property 9.2

- write

$$\mathbf{X} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \mathbf{e}_i \mathbf{e}_j^T$$

- by letting $\mathbf{a}_i$ be the ith column of $\mathbf{A}$ and $\mathbf{b}_j$ the jth row of $\mathbf{B}$,

$$\mathrm{vec}(\mathbf{AXB}) = \mathrm{vec}\left(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \mathbf{A} \mathbf{e}_i \mathbf{e}_j^T \mathbf{B} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \mathrm{vec}(\mathbf{a}_i \mathbf{b}_j^T).$$

- by noting

$$\mathrm{vec}(\mathbf{a}_i \mathbf{b}_j^T) = \mathrm{vec}([\ \mathbf{a}_i b_{j1}, \ldots, \mathbf{a}_i b_{j,q}\ ]) = \begin{bmatrix} b_{j1}\mathbf{a}_i \\ \vdots \\ b_{j,q}\mathbf{a}_i \end{bmatrix} = \mathbf{b}_j \otimes \mathbf{a}_i$$

we get $\mathrm{vec}(\mathbf{AXB}) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \mathbf{b}_j \otimes \mathbf{a}_i = (\mathbf{B}^T \otimes \mathbf{A}) \mathrm{vec}(\mathbf{X})$.

Kronecker Sum

- Problem: given $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{m \times m}$, $\mathbf{C} \in \mathbb{R}^{n \times m}$, solve

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C} \qquad (*)$$

  with respect to $\mathbf{X} \in \mathbb{R}^{m \times n}$.

- the above problem is a linear system. By vectorizing $(*)$, we get

$$(\mathbf{I}_m \otimes \mathbf{A})\text{vec}(\mathbf{X}) + (\mathbf{B}^T \otimes \mathbf{I}_n)\text{vec}(\mathbf{X}) = \text{vec}(\mathbf{C})$$

- the Kronecker sum of $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times m}$ is

$$\mathbf{A} \oplus \mathbf{B} = (\mathbf{I}_m \otimes \mathbf{A}) + (\mathbf{B} \otimes \mathbf{I}_n).$$

  – if a unique solution to $(*)$ is desired, we wish to know conditions under which $\mathbf{A} \oplus \mathbf{B}$ is nonsingular

Kronecker Sum

Theorem 9.2. Let $\{\lambda_i, \mathbf{x}_i\}_{i=1}^n$ be the set of n eigen-pairs of $\mathbf{A}$, and let $\{\mu_i, \mathbf{y}_i\}_{i=1}^m$ be the set of m eigen-pairs of $\mathbf{B}$. The set of mn eigen-pairs of $\mathbf{A} \oplus \mathbf{B}$ is given by

$$\{\lambda_i + \mu_j, \mathbf{y}_j \otimes \mathbf{x}_i\}_{i=1,\ldots,n,\ j=1,\ldots,m}$$

Theorem 9.3. The matrix equations

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C}$$

has a unique solution for every given $\mathbf{C}$ if and only if

$$\lambda_i \neq -\mu_j, \qquad \text{for all } i, j,$$

where $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^m$ are the set of eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, resp.

- idea behind Theorem 9.3: if $\lambda_i = -\mu_j$ for some i, j, then from Theorem 9.2 there exists a zero eigenvalue for $\mathbf{A} \oplus \mathbf{B}$.

Kronecker Sum

- Consider

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} = \mathbf{C},$$

  which is called the Lyapunov equations.

- from Theorem 9.3, the Lyapunov equations admit a unique solution if

$$\lambda_i \neq -\lambda_j, \qquad \text{for all } i,\ j.$$

- if $\mathbf{A}$ is PD such that $\lambda_i > 0$ for all i, the Lyapunov equations always have a unique solution.