# ENGG 5781 Matrix Analysis and Computations
# Lecture 5: Singular Value Decomposition

Wing-Kin (Ken) Ma

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Department of Electronic Engineering
The Chinese University of Hong Kong

# Lecture 5: Singular Value Decomposition

- singular value decomposition
- matrix norms
- linear systems
- LS, pseudo-inverse, orthogonal projections
- low-rank matrix approximation
- singular value inequalities
- computing the SVD via the power method

# Main Results

- any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ admits a singular value decomposition

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T,$$

  where $\mathbf{U} \in \mathbb{R}^{m\times m}$ and $\mathbf{V} \in \mathbb{R}^{m\times m}$ are orthogonal, and $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times n}$ has $[\boldsymbol{\Sigma}]_{ij} = 0$ for all $i \neq j$ and $[\boldsymbol{\Sigma}]_{ii} = \sigma_i$ for all $i$, with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min\{m,n\}}$.

- matrix 2-norm: $\|\mathbf{A}\|_2 = \sigma_1$

- let $r$ be the number of nonzero $\sigma_i$'s, partition $\mathbf{U} = [\ \mathbf{U}_1\ \mathbf{U}_2\ ]$, $\mathbf{V} = [\ \mathbf{V}_1\ \mathbf{V}_2\ ]$, and let $\tilde{\boldsymbol{\Sigma}} = \mathrm{Diag}(\sigma_1, \ldots, \sigma_r)$
  - pseudo-inverse: $\mathbf{A}^\dagger = \mathbf{V}_1\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{U}_1^T$
  - LS solution: $\mathbf{x}_{\mathsf{LS}} = \mathbf{A}^\dagger\mathbf{y} + \boldsymbol{\eta}$ for any $\boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2)$
  - orthogonal projection: $\mathbf{P}_{\mathbf{A}} = \mathbf{U}_1\mathbf{U}_1^T$

## Main Results

- low-rank matrix approximation: given $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $k \in \{1,\ldots,\min\{m,n\}\}$, the problem

$$\min_{\mathbf{B}\in\mathbb{R}^{m\times n},\ \mathrm{rank}(\mathbf{B})\leq k} \|\mathbf{A}-\mathbf{B}\|_F^2$$

has a solution given by $\mathbf{B}^\star = \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^T$

# Singular Value Decomposition

**Theorem 5.1.** Given any $\mathbf{A} \in \mathbb{R}^{m\times n}$, there exists a 3-tuple $(\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n}$ such that

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T,$$

$\mathbf{U}$ and $\mathbf{V}$ are orthogonal, and $\mathbf{\Sigma}$ takes the form

$$[\mathbf{\Sigma}]_{ij} = \begin{cases} \sigma_i, & i = j \\ 0, & i \neq j \end{cases}, \qquad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0,\ p = \min\{m, n\}.$$

- the above decomposition is called the singular value decomposition (SVD)
- $\sigma_i$ is called the $i$th singular value
- $\mathbf{u}_i$ and $\mathbf{v}_i$ are called the $i$th left and right singular vectors, resp.
- the following notations may be used to denote singular values of a given $\mathbf{A}$

$$\sigma_{\max}(\mathbf{A}) = \sigma_1(\mathbf{A}) \geq \sigma_2(\mathbf{A}) \geq \ldots \geq \sigma_p(\mathbf{A}) = \sigma_{\min}(\mathbf{A})$$

# Different Ways of Writing out SVD

- partitioned form: let $r$ be the number of nonzero singular values, and note $\sigma_1 \geq \ldots \sigma_r > 0$, $\sigma_{r+1} = \ldots = \sigma_p = 0$. Then,

$$\mathbf{A} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\Sigma}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^T \\ \mathbf{V}_2^T \end{bmatrix},$$

  where

  - $\tilde{\boldsymbol{\Sigma}} = \mathrm{Diag}(\sigma_1, \ldots, \sigma_r)$,
  - $\mathbf{U}_1 = [\ \mathbf{u}_1, \ldots, \mathbf{u}_r\ ] \in \mathbb{R}^{m \times r}$, $\mathbf{U}_2 = [\ \mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\ ] \in \mathbb{R}^{m \times (m-r)}$,
  - $\mathbf{V}_1 = [\ \mathbf{v}_1, \ldots, \mathbf{v}_r\ ] \in \mathbb{R}^{n \times r}$, $\mathbf{V}_2 = [\ \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\ ] \in \mathbb{R}^{n \times (n-r)}$.

- thin SVD: $\mathbf{A} = \mathbf{U}_1 \tilde{\boldsymbol{\Sigma}} \mathbf{V}_1^T$

- outer-product form: $\mathbf{A} = \sum_{i=1}^{p} \sigma_i \mathbf{u}_i \mathbf{v}_i^T = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$

# SVD and Eigendecomposition

From the SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, we see that

$$\mathbf{A}\mathbf{A}^T = \mathbf{U}\mathbf{D}_1\mathbf{U}^T, \qquad \mathbf{D}_1 = \mathbf{\Sigma}\mathbf{\Sigma}^T = \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_p^2, \underbrace{0, \ldots, 0}_{m-p \text{ zeros}}) \tag{$*$}$$

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\mathbf{D}_2\mathbf{V}^T, \qquad \mathbf{D}_2 = \mathbf{\Sigma}^T\mathbf{\Sigma} = \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_p^2, \underbrace{0, \ldots, 0}_{n-p \text{ zeros}}) \tag{$**$}$$

Observations:

- $(*)$ and $(**)$ are the eigendecompositions of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$, resp.
- the left singular matrix $\mathbf{U}$ of $\mathbf{A}$ is the eigenvector matrix of $\mathbf{A}\mathbf{A}^T$
- the right singular matrix $\mathbf{V}$ of $\mathbf{A}$ is the eigenvector matrix of $\mathbf{A}^T\mathbf{A}$
- the squares of nonzero singular values of $\mathbf{A}$, $\sigma_1^2, \ldots, \sigma_r^2$, are the nonzero eigenvalues of both $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$.

# Insights of the Proof of SVD

- the proof of SVD is constructive
- to see the insights, consider the special case of square nonsingular $\mathbf{A}$
- $\mathbf{A}\mathbf{A}^T$ is PD, and denote its eigendecomposition by

$$\mathbf{A}\mathbf{A}^T = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T, \qquad \text{with } \lambda_1 \geq \ldots \geq \lambda_n > 0.$$

- let $\boldsymbol{\Sigma} = \mathrm{Diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m})$, $\mathbf{V} = \mathbf{A}^T\mathbf{U}\boldsymbol{\Sigma}^{-1}$
- it can be verified that $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \mathbf{A}$, $\mathbf{V}^T\mathbf{V} = \mathbf{I}$
- see the accompanying note for the proof of SVD in the general case

# SVD and Subspace

**Property 5.1.** The following properties hold:

(a) $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{U}_1)$, $\mathcal{R}(\mathbf{A})^\perp = \mathcal{R}(\mathbf{U}_2)$;

(b) $\mathcal{R}(\mathbf{A}^T) = \mathcal{R}(\mathbf{V}_1)$, $\mathcal{R}(\mathbf{A}^T)^\perp = \mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{V}_2)$;

(c) $\mathrm{rank}(\mathbf{A}) = r$ (the number of nonzero singular values).

Note:

- in practice, SVD can be used a numerical tool for computing bases of $\mathcal{R}(\mathbf{A})$, $\mathcal{R}(\mathbf{A})^\perp$, $\mathcal{R}(\mathbf{A}^T)$, $\mathcal{N}(\mathbf{A})$
- we have previously learnt the following properties
  - $\mathrm{rank}(\mathbf{A}^T) = \mathrm{rank}(\mathbf{A})$
  - $\dim \mathcal{N}(\mathbf{A}) = n - \mathrm{rank}(\mathbf{A})$

  By SVD, the above properties are easily seen to be true

# Matrix Norms

- the definition of a norm of a matrix is the same as that of a vector:
  - $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is a norm if (i) $f(\mathbf{A}) \geq 0$ for all $\mathbf{A}$; (ii) $f(\mathbf{A}) = 0$ if and only if $\mathbf{A} = \mathbf{0}$; (iii) $f(\mathbf{A} + \mathbf{B}) \leq f(\mathbf{A}) + f(\mathbf{B})$ for any $\mathbf{A}, \mathbf{B}$; (iv) $f(\alpha \mathbf{A}) = |\alpha| f(\mathbf{A})$ for any $\alpha, \mathbf{A}$
- naturally, the Frobenius norm $\|\mathbf{A}\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = [\mathrm{tr}(\mathbf{A}^T \mathbf{A})]^{1/2}$ is a norm
- there are many other matrix norms
- induced norm or operator norm: the function

$$f(\mathbf{A}) = \max_{\|\mathbf{x}\|_\beta \leq 1} \|\mathbf{A}\mathbf{x}\|_\alpha$$

where $\| \cdot \|_\alpha, \| \cdot \|_\beta$ denote any vector norms, can be shown be to a norm

# Matrix Norms

- matrix norms induced by the vector $p$-norm ($p \geq 1$):

$$\|\mathbf{A}\|_p = \max_{\|\mathbf{x}\|_p \leq 1} \|\mathbf{A}\mathbf{x}\|_p$$

- it is known that
  - $\|\mathbf{A}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$
  - $\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$

- how about $p = 2$?

# Matrix $2$-Norm

- matrix $2$-norm or spectral norm:

$$\|\mathbf{A}\|_2 = \sigma_{\max}(\mathbf{A}).$$

- proof:
  - for any $\mathbf{x}$ with $\|\mathbf{x}\|_2 \leq 1$,

$$\begin{aligned} \|\mathbf{A}\mathbf{x}\|_2^2 &= \|\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{x}\|_2^2 = \|\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{x}\|_2^2 \\ &\leq \sigma_1^2\|\mathbf{V}^T\mathbf{x}\|_2^2 = \sigma_1^2\|\mathbf{x}\|_2^2 \leq \sigma_1^2 \end{aligned}$$

  - $\|\mathbf{A}\mathbf{x}\|_2 = \sigma_1$ if we choose $\mathbf{x} = \mathbf{v}_1$

- implication to linear systems: let $\mathbf{y} = \mathbf{A}\mathbf{x}$ be a linear system. Under the input energy constraint $\|\mathbf{x}\|_2 \leq 1$, the system output energy $\|\mathbf{y}\|_2^2$ is maximized when $\mathbf{x}$ is chosen as the $1$st right singular vector

- corollary: $\displaystyle\min_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \sigma_{\min}(\mathbf{A})$ if $m \geq n$

# Matrix $2$-Norm

Properties for the matrix $2$-norm:

- $\|\mathbf{AB}\|_2 \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2$
  - in fact, $\|\mathbf{AB}\|_p \leq \|\mathbf{A}\|_p \|\mathbf{B}\|_p$ for any $p \geq 1$
- $\|\mathbf{Ax}\|_2 \leq \|\mathbf{A}\|_2 \|\mathbf{x}\|_2$
  - a special case of the 1st property
- $\|\mathbf{QAW}\|_2 = \|\mathbf{A}\|_2$ for any orthogonal $\mathbf{Q}, \mathbf{W}$
  - we also have $\|\mathbf{QAW}\|_F = \|\mathbf{A}\|_F$ for any orthogonal $\mathbf{Q}, \mathbf{W}$
- $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F \leq \sqrt{p}\|\mathbf{A}\|_2$ (here $p = \min\{m, n\}$)
  - proof: $\|\mathbf{A}\|_F = \|\mathbf{\Sigma}\|_F = \sqrt{\sum_{i=1}^p \sigma_i^2}$, and $\sigma_1^2 \leq \sum_{i=1}^p \sigma_i^2 \leq p\sigma_1^2$

# Schatten $p$-Norm

- the function

$$f(\mathbf{A}) = \left( \sum_{i=1}^{\min\{m,n\}} \sigma_i(\mathbf{A})^p \right)^{1/p}, \qquad p \geq 1,$$

  is known to be a norm and is called the Schatten $p$-norm

- nuclear norm:

$$\|\mathbf{A}\|_* = \sum_{i=1}^{\min\{m,n\}} \sigma_i(\mathbf{A})$$

  – a special case of the Schatten $p$-norm

  – a way to prove that the nuclear norm is a norm:

    ∗ show that $f(\mathbf{A}) = \max_{\|\mathbf{B}\|_2 \leq 1} \operatorname{tr}(\mathbf{B}^T \mathbf{A})$ is a norm

    ∗ show that $f(\mathbf{A}) = \sum_{i=1}^{\min\{m,n\}} \sigma_i$

  – finds applications in rank approximation, e.g., for compressive sensing and matrix completion **[Recht-Fazel-Parrilo'10]**

# Schatten $p$-Norm

- $\mathrm{rank}(\mathbf{A})$ is nonconvex in $\mathbf{A}$ and is arguably hard to do optimization with it
- Idea: the rank function can be expressed as

$$\mathrm{rank}(\mathbf{A}) = \sum_{i=1}^{\min\{m,n\}} \mathbb{1}\{\sigma_i(\mathbf{A}) \neq 0\},$$

  and why not approximate it by

$$f(\mathbf{A}) = \textstyle\sum_{i=1}^{\min\{m,n\}} \varphi(\sigma_i(\mathbf{A}))$$

  for some friendly function $\varphi$?

- nuclear norm

$$\|\mathbf{A}\|_* = \textstyle\sum_{i=1}^{\min\{m,n\}} \sigma_i(\mathbf{A})$$

  - uses $\varphi(z) = z$
  - is convex in $\mathbf{A}$

# Linear Systems: Sensitivity Analysis

- Scenario:
  - let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be nonsingular, and $\mathbf{y} \in \mathbb{R}^n$. Let $\mathbf{x}$ be the solution to

$$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

  - consider a perturbed version of the above system: $\hat{\mathbf{A}} = \mathbf{A} + \Delta\mathbf{A}, \hat{\mathbf{y}} = \mathbf{y} + \Delta\mathbf{y}$, where $\Delta\mathbf{A}$ and $\Delta\mathbf{y}$ are errors. Let $\hat{\mathbf{x}}$ be a solution to the perturbed system

$$\hat{\mathbf{y}} = \hat{\mathbf{A}}\hat{\mathbf{x}}.$$

- Problem: analyze how the solution error $\|\hat{\mathbf{x}} - \mathbf{x}\|_2$ scales with $\Delta\mathbf{A}$ and $\Delta\mathbf{y}$

- remark: $\Delta\mathbf{A}$ and $\Delta\mathbf{y}$ may be floating point errors, measurement errors, etc

# Linear Systems: Sensitivity Analysis

- the condition number of a given matrix $\mathbf{A}$ is defined as

$$\kappa(\mathbf{A}) = \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})},$$

- $\kappa(\mathbf{A}) \geq 1$, and $\kappa(\mathbf{A}) = 1$ if $\mathbf{A}$ is orthogonal

- $\mathbf{A}$ is said to be ill-conditioned if $\kappa(\mathbf{A})$ is very large; that refers to cases where $\mathbf{A}$ is close to singular

# Linear Systems: Sensitivity Analysis

**Theorem 5.2.** Let $\varepsilon > 0$ be a constant such that

$$\frac{\|\Delta \mathbf{A}\|_2}{\|\mathbf{A}\|_2} \leq \varepsilon, \qquad \frac{\|\Delta \mathbf{y}\|_2}{\|\mathbf{y}\|_2} \leq \varepsilon.$$

If $\varepsilon$ is sufficiently small such that $\varepsilon\kappa(\mathbf{A}) < 1$, then

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \frac{2\varepsilon\kappa(\mathbf{A})}{1 - \varepsilon\kappa(\mathbf{A})}.$$

- Implications:
  - for small errors and in the worst-case sense, the relative error $\|\hat{\mathbf{x}} - \mathbf{x}\|_2 / \|\mathbf{x}\|_2$ tends to increase with the condition number
  - in particular, for $\varepsilon\kappa(\mathbf{A}) \leq \frac{1}{2}$, the error bound can be simplified to

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq 4\varepsilon\kappa(\mathbf{A})$$

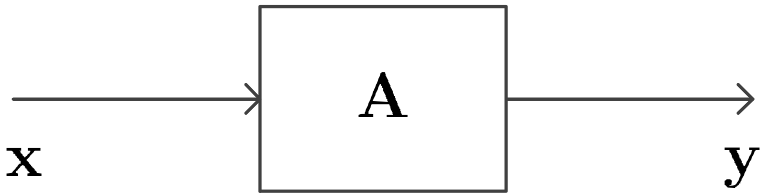
# Linear Systems: Interpretation under SVD

- consider the linear system
$$\mathbf{y} = \mathbf{A}\mathbf{x}$$
where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the system matrix; $\mathbf{x} \in \mathbb{R}^n$ is the system input; $\mathbf{y} \in \mathbb{R}^m$ is the system output

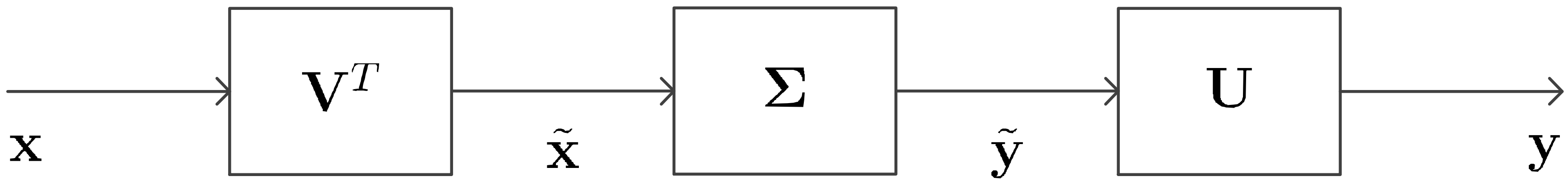
- by SVD we can write
$$\mathbf{y} = \mathbf{U}\tilde{\mathbf{y}}, \qquad \tilde{\mathbf{y}} = \mathbf{\Sigma}\tilde{\mathbf{x}}, \qquad \tilde{\mathbf{x}} = \mathbf{V}^T\mathbf{x}$$

- **Implication:** *all* linear systems work by performing three processes in cascade, namely,
  - rotate/reflect the system input $\mathbf{x}$ to form an intermediate system input $\tilde{\mathbf{x}}$
  - form an intermediate system output $\tilde{\mathbf{y}}$ by element-wise rescaling $\tilde{\mathbf{x}}$ w.r.t. $\sigma_i$'s and by either removing some entires of $\tilde{\mathbf{x}}$ or adding some zeros
  - rotate/reflect $\tilde{\mathbf{y}}$ to form the system output $\mathbf{y}$

# Linear Systems: Interpretation under SVD




(a) linear system




(b) equivalent system

# Linear Systems: Solution via SVD

- Problem: given *general* $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{y} \in \mathbb{R}^m$, determine
  - whether $\mathbf{y} = \mathbf{A}\mathbf{x}$ has a solution (more precisely, whether there exists an $\mathbf{x}$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}$);
  - what is the solution

- by SVD it can be shown that

$$
\begin{aligned}
\mathbf{y} = \mathbf{A}\mathbf{x} \quad & \Longleftrightarrow \quad \mathbf{y} = \mathbf{U}_1 \tilde{\mathbf{\Sigma}} \mathbf{V}_1^T \mathbf{x} \\
& \Longleftrightarrow \quad \mathbf{U}_1^T \mathbf{y} = \tilde{\mathbf{\Sigma}} \mathbf{V}_1^T \mathbf{x}, \; \mathbf{U}_2^T \mathbf{y} = \mathbf{0} \\
& \Longleftrightarrow \quad \mathbf{V}_1^T \mathbf{x} = \tilde{\mathbf{\Sigma}}^{-1} \mathbf{U}_1^T \mathbf{y}, \; \mathbf{U}_2^T \mathbf{y} = \mathbf{0} \\
& \Longleftrightarrow \quad \begin{array}{l} \mathbf{x} = \mathbf{V}_1 \tilde{\mathbf{\Sigma}}^{-1} \mathbf{U}_1^T \mathbf{y} + \boldsymbol{\eta}, \; \text{for any } \boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2) = \mathcal{N}(\mathbf{A}), \\ \mathbf{U}_2^T \mathbf{y} = \mathbf{0} \end{array}
\end{aligned}
$$

## Linear Systems: Solution via SVD

- let us consider specific cases of the linear system solution characterization

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad \Longleftrightarrow \quad \begin{array}{l} \mathbf{x} = \mathbf{V}_1 \tilde{\boldsymbol{\Sigma}}^{-1} \mathbf{U}_1^T \mathbf{y} + \boldsymbol{\eta}, \text{ for any } \boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2) = \mathcal{N}(\mathbf{A}), \\ \mathbf{U}_2^T \mathbf{y} = \mathbf{0} \end{array}$$

- Case (a): full-column rank $\mathbf{A}$, i.e., $r = n \leq m$
  - there is no $\mathbf{V}_2$, and $\mathbf{U}_2^T \mathbf{y} = \mathbf{0}$ is equivalent to $\mathbf{y} \in \mathcal{R}(\mathbf{U}_1) = \mathcal{R}(\mathbf{A})$
  - Result: the linear system has a solution if and only if $\mathbf{y} \in \mathcal{R}(\mathbf{A})$, and the solution, if exists, is uniquely given by $\mathbf{x} = \mathbf{V}\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{U}_1^T\mathbf{y}$

- Case (b): full-row rank $\mathbf{A}$, i.e., $r = m \leq n$
  - there is no $\mathbf{U}_2$
  - Result: the linear system always has a solution, and the solution is given by $\mathbf{x} = \mathbf{V}_1\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{U}^T\mathbf{y} + \boldsymbol{\eta}$ for any $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A})$

## Least Squares via SVD

- consider the LS problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2$$

for *general* $\mathbf{A}\in\mathbb{R}^{m\times n}$

- we have, for any $\mathbf{x}\in\mathbb{R}^n$,

$$\begin{aligned}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 &= \|\mathbf{y}-\mathbf{U}\boldsymbol{\Sigma}\underbrace{\mathbf{V}^T\mathbf{x}}_{=\tilde{\mathbf{x}}}\|_2^2 = \|\underbrace{\mathbf{U}^T\mathbf{y}}_{=\tilde{\mathbf{y}}}-\boldsymbol{\Sigma}\tilde{\mathbf{x}}\|_2^2 \\ &= \textstyle\sum_{i=1}^r |\tilde{y}_i-\sigma_i\tilde{x}_i|^2 + \sum_{i=r+1}^p |\tilde{y}_i|^2 \\ &\geq \textstyle\sum_{i=r+1}^p |\tilde{y}_i|^2\end{aligned}$$

- the equality above is attained if $\tilde{\mathbf{x}}$ satisfies $\tilde{y}_i=\sigma_i\tilde{x}_i$ for $i=1,\ldots,r$, and it can be shown that such a $\tilde{\mathbf{x}}$ corresponds to (try)

$$\mathbf{x}=\mathbf{V}_1\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{U}_1^T\mathbf{y}+\mathbf{V}_2\tilde{\mathbf{x}}_2, \quad \text{for any } \tilde{\mathbf{x}}_2\in\mathbb{R}^{n-r}$$

which is the desired LS solution

# Pseudo-Inverse

The pseudo-inverse of a matrix $\mathbf{A}$ is defined as

$$\mathbf{A}^\dagger = \mathbf{V}_1 \tilde{\boldsymbol{\Sigma}}^{-1} \mathbf{U}_1^T.$$

From the above def. we can show that

- $\mathbf{x}_{\mathsf{LS}} = \mathbf{A}^\dagger \mathbf{y} + \boldsymbol{\eta}$ for any $\boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2)$; the same applies to linear sys. $\mathbf{y} = \mathbf{A}\mathbf{x}$
- $\mathbf{A}^\dagger$ satisfies the Moore-Penrose conditions: (i) $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$; (ii) $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$; (iii) $\mathbf{A}\mathbf{A}^\dagger$ is symmetric; (iv) $\mathbf{A}^\dagger\mathbf{A}$ is symmetric
- when $\mathbf{A}$ has full column rank
  - the pseudo-inverse also equals $\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$
  - $\mathbf{A}^\dagger\mathbf{A} = \mathbf{I}$
- when $\mathbf{A}$ has full row rank
  - the pseudo-inverse also equals $\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$
  - $\mathbf{A}\mathbf{A}^\dagger = \mathbf{I}$

# Orthogonal Projections

- with SVD, the orthogonal projections of $\mathbf{y}$ onto $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})^\perp$ are, resp.,

$$\Pi_{\mathcal{R}(\mathbf{A})}(\mathbf{y}) = \mathbf{A}\mathbf{x}_{\mathsf{LS}} = \mathbf{A}\mathbf{A}^\dagger\mathbf{y} = \mathbf{U}_1\mathbf{U}_1^T\mathbf{y}$$

$$\Pi_{\mathcal{R}(\mathbf{A})^\perp}(\mathbf{y}) = \mathbf{y} - \mathbf{A}\mathbf{x}_{\mathsf{LS}} = (\mathbf{I} - \mathbf{A}\mathbf{A}^\dagger)\mathbf{y} = \mathbf{U}_2\mathbf{U}_2^T\mathbf{y}$$

- the orthogonal projector and orthogonal complement projector of $\mathbf{A}$ are resp. defined as

$$\mathbf{P}_\mathbf{A} = \mathbf{U}_1\mathbf{U}_1^T, \qquad \mathbf{P}_\mathbf{A}^\perp = \mathbf{U}_2\mathbf{U}_2^T$$

- properties (easy to show):
  - $\mathbf{P}_\mathbf{A}$ is idempotent, i.e., $\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{A} = \mathbf{P}_\mathbf{A}$
  - $\mathbf{P}_\mathbf{A}$ is symmetric
  - the eigenvalues of $\mathbf{P}_\mathbf{A}$ are either $0$ or $1$
  - $\mathcal{R}(\mathbf{P}_\mathbf{A}) = \mathcal{R}(\mathbf{A})$
  - the same properties above apply to $\mathbf{P}_\mathbf{A}^\perp$, and $\mathbf{I} = \mathbf{P}_\mathbf{A} + \mathbf{P}_\mathbf{A}^\perp$

## Minimum $2$-Norm Solution to Underdetermined Linear Systems

- consider solving the linear system $\mathbf{y} = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ is fat
- this is an underdetermined problem: we have more unknowns $n$ than the number of equations $m$
- assume that $\mathbf{A}$ has full row rank. By now we know that any

$$\mathbf{x} = \mathbf{A}^\dagger \mathbf{y} + \boldsymbol{\eta}, \quad \boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2)$$

  is a solution to $\mathbf{y} = \mathbf{A}\mathbf{x}$, but we may want to grab one solution only

- Idea: discard $\boldsymbol{\eta}$ and take $\mathbf{x} = \mathbf{A}^\dagger \mathbf{y}$ as our solution
- Question: does discarding $\boldsymbol{\eta}$ make sense?
- Answer: it makes sense under the minimum $2$-norm problem formulation

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_2^2 \qquad \text{s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}$$

  It can be shown that the solution is uniquely given by $\mathbf{x} = \mathbf{A}^\dagger \mathbf{y}$ (try the proof)

# Low-Rank Matrix Approximation

**Aim:** given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and an integer $k$ with $1 \leq k < \mathrm{rank}(\mathbf{A})$, find a matrix $\mathbf{B} \in \mathbb{R}^{m \times n}$ such that $\mathrm{rank}(\mathbf{B}) \leq k$ and $\mathbf{B}$ best approximates $\mathbf{A}$

- it is somehow unclear about what a best approximation means, and we will specify one later

- closely related to the matrix factorization problem considered in Lecture 2

- applications: PCA, dimensionality reduction,...—the same kind of applications in matrix factorization

- truncated SVD: denote
$$\mathbf{A}_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T.$$
Perform the aforementioned approximation by choosing $\mathbf{B} = \mathbf{A}_k$

# Toy Application Example: Image Compression

- let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix whose $(i,j)$th entry $a_{ij}$ stores the $(i,j)$th pixel of an image.

- memory size for storing $\mathbf{A}$: $mn$

- truncated SVD: store $\{\mathbf{u}_i, \sigma_i \mathbf{v}_i\}_{i=1}^k$ instead of the full $\mathbf{A}$, and recover the image by $\mathbf{B} = \mathbf{A}_k$

- memory size for truncated SVD: $(m+n)k$
  - much less than $mn$ if $k \ll \min\{m,n\}$

# Toy Application Example: Image Compression

(a) original image, size= $102 \times 1347$


(b) truncated SVD, k= 5

(c) truncated SVD, k= 10


(d) truncated SVD, k= 20

# Low-Rank Matrix Approximation

- truncated SVD provides the best approximation in the LS sense:

  **Theorem 5.3** (Eckart-Young-Mirsky)**.** Consider the following problem

  $$\min_{\mathbf{B}\in\mathbb{R}^{m\times n},\ \mathrm{rank}(\mathbf{B})\leq k} \|\mathbf{A}-\mathbf{B}\|_F^2$$

  where $\mathbf{A}\in\mathbb{R}^{m\times n}$ and $k\in\{1,\ldots,p\}$ are given. The truncated SVD $\mathbf{A}_k$ is an optimal solution to the above problem.

- also note the matrix $2$-norm version of the Eckart-Young-Mirsky theorem:

  **Theorem 5.4.** Consider the following problem

  $$\min_{\mathbf{B}\in\mathbb{R}^{m\times n},\ \mathrm{rank}(\mathbf{B})\leq k} \|\mathbf{A}-\mathbf{B}\|_2$$

  where $\mathbf{A}\in\mathbb{R}^{m\times n}$ and $k\in\{1,\ldots,p\}$ are given. The truncated SVD $\mathbf{A}_k$ is an optimal solution to the above problem.

## Low-Rank Matrix Approximation

- recall the matrix factorization problem in Lecture 2:

$$\min_{\mathbf{A}\in\mathbb{R}^{m\times k},\mathbf{B}\in\mathbb{R}^{k\times n}} \|\mathbf{Y}-\mathbf{A}\mathbf{B}\|_F^2$$

  where $k \leq \min\{m,n\}$; $\mathbf{A}$ denotes a basis matrix; $\mathbf{B}$ is the coefficient matrix

- the matrix factorizaton problem may be reformulated as (verify)

$$\min_{\mathbf{Z}\in\mathbb{R}^{m\times n},\mathrm{rank}(\mathbf{Z})\leq k} \|\mathbf{Y}-\mathbf{Z}\|_F^2,$$

  and the truncated SVD $\mathbf{Y}_k = \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^T$, where $\mathbf{Y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ denotes the SVD of $\mathbf{Y}$, is an optimal solution by Theorem 5.4

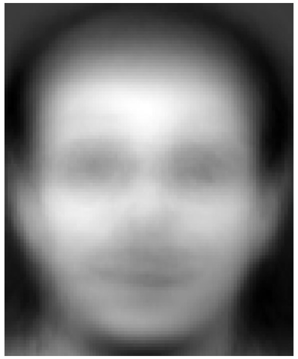
- thus, an optimal solution to the matrix factorization problem is

$$\mathbf{A} = [\ \mathbf{u}_1,\ldots,\mathbf{u}_k\ ], \qquad \mathbf{B} = [\ \sigma_1\mathbf{v}_1,\ldots,\sigma_k\mathbf{v}_k\ ]^T$$

# Toy Demo: Dimensionality Reduction of a Face Image Dataset

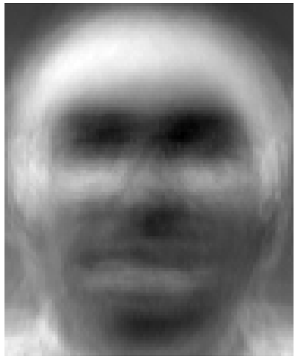
A face image dataset. Image size $= 112 \times 92$, number of face images $= 400$. Each $\mathbf{x}_i$ is the vectorization of one face image, leading to $m = 112 \times 92 = 10304$, $n = 400$.

# Toy Demo: Dimensionality Reduction of a Face Image Dataset

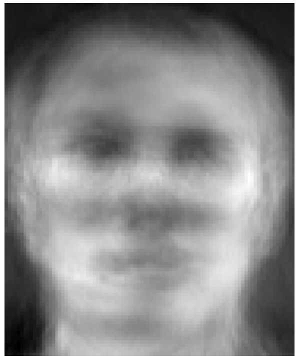


Mean face

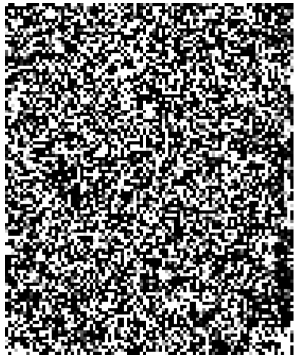
1st principal left singular vector



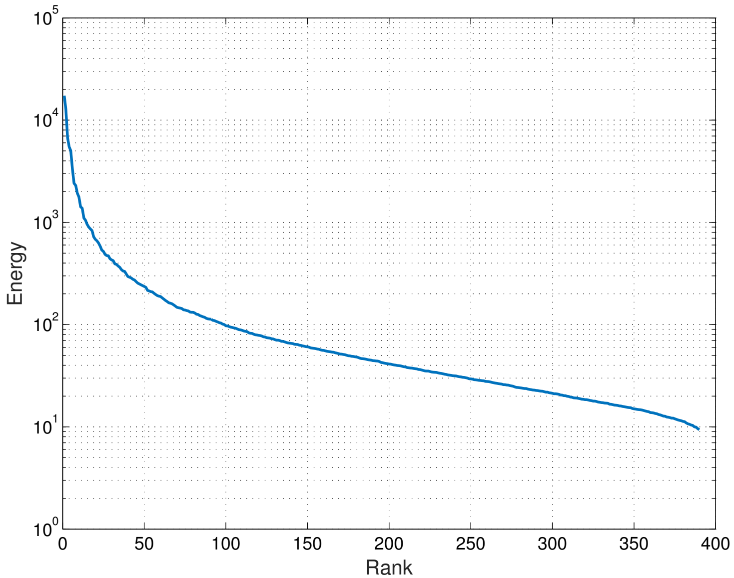
2nd principal left singular vector



3rd principal left singular vector



400th left singular vector




Energy Concentration

# Singular Value Inequalities

Similar to variational characterization of eigenvalues of real symmetric matrices, we can derive various variational characterization results for singular values, e.g.,

- Courant-Fischer characterization:

$$\sigma_k(\mathbf{A}) = \min_{\dim \mathcal{S}_{n-k+1} \subseteq \mathbb{R}^n} \; \max_{\mathbf{x} \in \mathcal{S}_{n-k+1}, \; \|\mathbf{x}\|_2 = 1} \|\mathbf{A}\mathbf{x}\|_2$$

- Weyl's inequality: given $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$,

$$\sigma_{k+l-1}(\mathbf{A}+\mathbf{B}) \leq \sigma_k(\mathbf{A}) + \sigma_l(\mathbf{B}), \qquad k, l \in \{1, \ldots, p\}, \; k + l - 1 \leq p.$$

  Note the special case

$$\sigma_k(\mathbf{A}) - \sigma_1(\mathbf{B}) \leq \sigma_k(\mathbf{A}+\mathbf{B}) \leq \sigma_k(\mathbf{A}) + \sigma_1(\mathbf{B}), \qquad k \in \{1, \ldots, p\}.$$

## Singular Value Inequalities

- Von Neumann trace inequality: given $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$,

$$\sum_{i=1}^{p} \sigma_i(\mathbf{A})\sigma_{n-i+1}(\mathbf{B}) \leq \mathrm{tr}(\mathbf{A}^T\mathbf{B}) \leq \sum_{i=1}^{p} \sigma_i(\mathbf{A})\sigma_i(\mathbf{B})$$

- and many more...

# Proof of the Eckart-Young-Mirsky Thm. by Weyl's Inequality

An application of singular value inequalities is that of proving Theorem 5.4:

- for any $\mathbf{B}$ with $\mathrm{rank}(\mathbf{B}) \leq k$, we have
  - $\sigma_l(\mathbf{B}) = 0$ for $l > k$
  - (Weyl) $\sigma_{i+k}(\mathbf{A}) \leq \sigma_i(\mathbf{A}-\mathbf{B}) + \sigma_{k+1}(\mathbf{B}) = \sigma_i(\mathbf{A}-\mathbf{B})$ for $i = 1, \ldots, p-k$
  - and consequently

$$\|\mathbf{A}-\mathbf{B}\|_F^2 = \sum_{i=1}^{p} \sigma_i(\mathbf{A}-\mathbf{B})^2 \geq \sum_{i=1}^{p-k} \sigma_i(\mathbf{A}-\mathbf{B})^2 \geq \sum_{i=k+1}^{p} \sigma_i(\mathbf{A})^2$$

- the equality above is attained if we choose $\mathbf{B} = \mathbf{A}_k$

# Computing the SVD via the Power Method

The power method can be used to compute the thin SVD, and the idea is as follows.

- assume $m \geq n$ and $\sigma_1 > \sigma_2 > \ldots \sigma_n > 0$
- apply the power method to $\mathbf{A}^T\mathbf{A}$ to obtain $\mathbf{v}_1$
- obtain $\mathbf{u}_1 = \mathbf{A}\mathbf{v}_1/\|\mathbf{A}\mathbf{v}_1\|_2, \sigma_1 = \|\mathbf{A}\mathbf{v}_1\|_2$ (why is this true?)
- do deflation $\mathbf{A} := \mathbf{A} - \sigma_1\mathbf{u}_1\mathbf{v}_1^T$, and repeat the above steps until all singular components are found

# References

**[Recht-Fazel-Parrilo'10]** B. Recht, M. Fazel, and P. A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," *SIAM Review*, vol. 52, no. 3, pp. 471–501, 2010.