ENGG 5781: Matrix Analysis and Computations **2022-23 First Term**

Lecture 5: Singular Value Decomposition

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In this note we give the detailed proof of some results in the main slides.

1 Proof of SVD

Recall the SVD theorem:

Theorem 5.1 *Given any* $\mathbf{A} \in \mathbb{R}^{m \times n}$*, there exists a 3-tuple* $(\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times n} \times \mathbb{R}^{n \times n}$ *such that*

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T,$$

$\mathbf{U}$ *and* $\mathbf{V}$ *are orthogonal, and* $\mathbf{\Sigma}$ *takes the form*

$$[\mathbf{\Sigma}]_{ij} = \begin{cases} \sigma_i, & i = j \\ 0, & i \neq j \end{cases},$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$ *and with* $p = \min\{m, n\}$.

The proof is as follows. First, consider the matrix product $\mathbf{A}\mathbf{A}^T$. Since $\mathbf{A}\mathbf{A}^T$ is real symmetric and PSD, by eigendecomposition we can express $\mathbf{A}\mathbf{A}^T$ as

$$\mathbf{A}\mathbf{A}^T = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Lambda}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1^T \\ \mathbf{U}_2^T \end{bmatrix} = \mathbf{U}_1 \tilde{\mathbf{\Lambda}} \mathbf{U}_1^T, \tag{1}$$

where we assume that the eigenvalues are ordered such that $\lambda_1 \geq \ldots \geq \lambda_r > 0$ and $\lambda_{r+1} = \ldots = \lambda_p = 0$, with r being the number of nonzero eigenvalues; $\mathbf{U} \in \mathbb{R}^{m \times m}$ denotes a corresponding orthogonal eigenvector matrix; we partiton $\mathbf{U}$ as $\mathbf{U} = [\ \mathbf{U}_1\ \mathbf{U}_2\ ]$, with $\mathbf{U}_1 \in \mathbb{R}^{m \times r}$ and $\mathbf{U}_2 \in \mathbb{R}^{m \times (m-r)}$; $\tilde{\mathbf{\Lambda}} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_r)$. It is easy to verify from the decomposition above that

$$\mathbf{U}_2^T \mathbf{A} = \mathbf{0}. \tag{2}$$

To see this, we note from $\mathbf{U}_2^T \mathbf{U}_1 = \mathbf{0}$ that $\mathbf{U}_2^T \mathbf{A} (\mathbf{U}_2^T \mathbf{A})^T = \mathbf{0}$. By the simple result that $\mathbf{B}\mathbf{B}^T = \mathbf{0}$ implies $\mathbf{B} = \mathbf{0}$ (which is easy to show and whose proof is omitted here), we conclude that $\mathbf{U}_2^T \mathbf{A} = \mathbf{0}$.

Second, construct the following matrices

$$\tilde{\mathbf{\Sigma}} = \tilde{\mathbf{\Lambda}}^{1/2} = \mathrm{Diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}), \qquad \mathbf{V}_1 = \mathbf{A}^T \mathbf{U}_1 \tilde{\mathbf{\Sigma}}^{-1} \in \mathbb{R}^{n \times r}.$$

One can easily see from (1) that

$$\mathbf{V}_1^T \mathbf{V}_1 = \mathbf{I}.$$

Furthermore, let $\mathbf{V}_2 \in \mathbb{R}^{n \times (n-r)}$ be a matrix such that $\mathbf{V} = [\ \mathbf{V}_1\ \mathbf{V}_2\ ]$ is orthogonal; we know from Lecture 1 that such a matrix always exists. It can be verified that

$$\mathbf{U}_1^T \mathbf{A} \mathbf{V}_1 = \tilde{\mathbf{\Sigma}}, \qquad \mathbf{U}_1^T \mathbf{A} \mathbf{V}_2 = \mathbf{0}. \tag{3}$$

Third, consider the matrix product $\mathbf{U}^T\mathbf{A}\mathbf{V}$. We have

$$\begin{aligned}\mathbf{U}^T\mathbf{A}\mathbf{V} &= \begin{bmatrix}\mathbf{U}_1^T\mathbf{A}\mathbf{V}_1 & \mathbf{U}_1^T\mathbf{A}\mathbf{V}_2 \\ \mathbf{U}_2^T\mathbf{A}\mathbf{V}_1 & \mathbf{U}_2^T\mathbf{A}\mathbf{V}_2\end{bmatrix} \\ &= \begin{bmatrix}\tilde{\mathbf{\Sigma}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}.\end{aligned}$$

where (2) and (3) have been used. By multiplying the above equation on the left by $\mathbf{U}$ and on the right by $\mathbf{V}^T$, we obtain the desired result $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$. The proof is complete.

2 Sensitivity Analysis of the Linear System Solution

Recall the perturbed linear system problem: Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be nonsingular and $\mathbf{y} \in \mathbb{R}^n$, and denote $\mathbf{x}$ as the solution to the linear system $\mathbf{y} = \mathbf{A}\mathbf{x}$. The actual $\mathbf{A}$ and $\mathbf{y}$ we deal with are perturbed. To be specific, we have

$$\hat{\mathbf{A}} = \mathbf{A} + \Delta\mathbf{A}, \qquad \hat{\mathbf{y}} = \mathbf{y} + \Delta\mathbf{y},$$

where $\Delta\mathbf{A}$ and $\Delta\mathbf{y}$ are errors. Let $\hat{\mathbf{x}}$ denote a solution to the perturbed linear system

$$\hat{\mathbf{y}} = \hat{\mathbf{A}}\hat{\mathbf{x}}.$$

The problem is to analyze how the solution error $\hat{\mathbf{x}} - \mathbf{x}$ scales with $\Delta\mathbf{A}$ and $\Delta\mathbf{y}$.

This analysis problem can be tackled via SVD. To put into context, define

$$\kappa(\mathbf{A}) = \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})},$$

which is called the *condition number* of $\mathbf{A}$. Note that if $\mathbf{A}$ is close to singular, then $\sigma_{\min}(\mathbf{A})$ will be very small and we would expect a very large $\kappa(\mathbf{A})$. We have the following result.

Theorem 5.2 *Let $\varepsilon > 0$ be a constant such that*

$$\frac{\|\Delta\mathbf{A}\|_2}{\|\mathbf{A}\|_2} \leq \varepsilon, \qquad \frac{\|\Delta\mathbf{y}\|_2}{\|\mathbf{y}\|_2} \leq \varepsilon.$$

If ε is sufficiently small such that $\varepsilon\kappa(\mathbf{A}) < 1$, then

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \frac{2\varepsilon\kappa(\mathbf{A})}{1 - \varepsilon\kappa(\mathbf{A})}.$$

Theorem 5.2 suggests that for a sufficiently small error level ε, the relative solution error $\|\hat{\mathbf{x}} - \mathbf{x}\|_2/\|\mathbf{x}\|_2$ tends to increase with the condition number $\kappa(\mathbf{A})$. In particular, if $\varepsilon\kappa(\mathbf{A}) \leq 1/2$, we may simplify the relative solution error bound in Theorem 5.2 to

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq 4\varepsilon\kappa(\mathbf{A}),$$

where we can see that the error bound above scales linearly with $\kappa(\mathbf{A})$.

Proof of Theorem 5.2: For notational convenience, denote $\Delta\mathbf{x} = \hat{\mathbf{x}} - \mathbf{x}$. The perturbed linear system can be written as

$$(\mathbf{A} + \Delta\mathbf{A})(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{y} + \Delta\mathbf{y}.$$

The above equation can be re-organized as

$$\mathbf{A}\Delta\mathbf{x} = \Delta\mathbf{y} - \Delta\mathbf{A}\mathbf{x} - \Delta\mathbf{A}\Delta\mathbf{x},$$

and then

$$\Delta\mathbf{x} = \mathbf{A}^{-1}(\Delta\mathbf{y} - \Delta\mathbf{A}\mathbf{x} - \Delta\mathbf{A}\Delta\mathbf{x}).$$

Let us take 2-norm on the above equation:

$$\begin{aligned} \|\Delta\mathbf{x}\|_2 &\leq \|\mathbf{A}^{-1}\|_2 \|\Delta\mathbf{y} - \Delta\mathbf{A}\mathbf{x} - \Delta\mathbf{A}\Delta\mathbf{x}\|_2 \\ &\leq \|\mathbf{A}^{-1}\|_2 (\|\Delta\mathbf{y}\|_2 + \|\Delta\mathbf{A}\mathbf{x}\|_2 + \|\Delta\mathbf{A}\Delta\mathbf{x}\|_2) \\ &\leq \|\mathbf{A}^{-1}\|_2 (\|\Delta\mathbf{y}\|_2 + \|\Delta\mathbf{A}\|_2\|\mathbf{x}\|_2 + \|\Delta\mathbf{A}\|_2\|\Delta\mathbf{x}\|_2) \end{aligned} \tag{4}$$

where we have used the norm inequality $\|\mathbf{A}\mathbf{x}\|_2 \leq \|\mathbf{A}\|_2\|\mathbf{x}\|_2$ and the triangle inequality $\|\mathbf{x}+\mathbf{y}\|_2 \leq \|\mathbf{y}\|_2 + \|\mathbf{y}\|_2$ to obtain (4).

Next, we apply the assumptions $\|\Delta\mathbf{A}\|_2/\|\mathbf{A}\|_2 \leq \varepsilon$ and $\|\Delta\mathbf{y}\|_2/\|\mathbf{y}\|_2 \leq \varepsilon$ to (4). We have

$$\|\Delta\mathbf{y}\|_2 \leq \varepsilon\|\mathbf{y}\|_2 = \varepsilon\|\mathbf{A}\mathbf{x}\|_2 \leq \varepsilon\|\mathbf{A}\|_2\|\mathbf{x}\|_2, \tag{5}$$

and substituting (5) and $\|\Delta\mathbf{A}\|_2 \leq \varepsilon\|\mathbf{A}\|_2$ into (4) results in

$$\begin{aligned} \|\Delta\mathbf{x}\|_2 &\leq \|\mathbf{A}^{-1}\|_2\|\mathbf{A}\|_2(2\varepsilon\|\mathbf{x}\|_2 + \varepsilon\|\Delta\mathbf{x}\|_2) \\ &= 2\varepsilon\kappa(\mathbf{A})\|\mathbf{x}\|_2 + \varepsilon\kappa(\mathbf{A})\|\Delta\mathbf{x}\|_2, \end{aligned}$$

where the result $\|\mathbf{A}^{-1}\|_2 = \max_{i=1,\ldots,n} 1/\sigma_i(\mathbf{A}) = 1/\sigma_{\min}(\mathbf{A})$ has been used. The above inequality can be rewritten as

$$(1 - \varepsilon\kappa(\mathbf{A}))\|\Delta\mathbf{x}\|_2 \leq 2\varepsilon\kappa(\mathbf{A})\|\mathbf{x}\|_2,$$

and if $1 - \varepsilon\kappa(\mathbf{A}) > 0$ we can further rewrite

$$\frac{\|\Delta\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \frac{2\varepsilon\kappa(\mathbf{A})}{1 - \varepsilon\kappa(\mathbf{A})},$$

as desired. ■