ENGG5781 Matrix Analysis and Computations
Lecture 3: Eigenvalues and Eigenvectors

Wing-Kin (Ken) Ma

2022-23 First Term

Department of Electronic Engineering
The Chinese University of Hong Kong

Lecture 3: Eigenvalues and Eigenvectors

- facts about eigenvalues and eigenvectors
- eigendecomposition, the case of Hermitian and real symmetric matrices
- power method
- Schur decomposition
- PageRank: a case study

Notation and Conventions

- a square matrix $\mathbf{A}$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i, j with $i \neq j$, or equivalently, if $\mathbf{A}^T = \mathbf{A}$
  - example:

$$\mathbf{A} = \begin{bmatrix} 1 & -0.5 & 3 \\ -0.5 & -2 & 0.9 \\ 3 & 0.9 & 0.1 \end{bmatrix}$$

- a square matrix $\mathbf{A}$ is said to be Hermitian if $a_{ij} = a_{ji}^*$ for all i, j with $i \neq j$, or equivalently, if $\mathbf{A}^H = \mathbf{A}$
- we denote the set of all $n \times n$ real symmetric matrices by $\mathbb{S}^n$
- we denote the set of all $n \times n$ complex Hermitian matrices by $\mathbb{H}^n$

Notation and Conventions

- note the following subtleties:
  - by definition, a real symmetric matrix is also Hermitian
  - when we say that a matrix is Hermitian, we often imply that the matrix may be complex (at least for this course); a real Hermitian matrix is simply real symmetric
  - we can have a complex symmetric matrix, though we will not study it

Main Results

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) is said to admit an eigendecomposition if there exists a nonsingular $\mathbf{V} \in \mathbb{C}^{n \times n}$ and a collection of scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1},$$

where $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$.

- the above $(\mathbf{V}, \mathbf{\Lambda})$ satisfies $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, \ldots, n$, which are eigen-equations
- $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are required to be linearly independent
- eigendecomposition *does not* always exist

Main Results

A real symmetric matrix $\mathbf{A} \in \mathbb{S}^n$ always admits an eigendecomposition

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

where $\mathbf{V} \in \mathbb{R}^{n\times n}$ is orthogonal; $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all i.

A Hermitian matrix $\mathbf{A} \in \mathbb{H}^n$ always admits an eigendecomposition

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$$

where $\mathbf{V} \in \mathbb{C}^{n\times n}$ is unitary; $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all i.

- differences: a Hermitian or real symmetric matrix always has
  - an eigendecomposition
  - real λ_i's
  - a $\mathbf{V}$ that is not only nonsingular but also unitary

Eigenvalues and Eigenvectors

We start with the basic definition of eigenvalues and eigenvectors.

Problem: given a $\mathbf{A} \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$), find a vector $\mathbf{v} \in \mathbb{C}^n$ with $\mathbf{v} \neq \mathbf{0}$ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}, \qquad \text{for some } \lambda \in \mathbb{C} \qquad (*)$$

- $(*)$ is called an eigenvalue problem or eigen-equation
- let $(\mathbf{v}, \lambda)$ be a solution to $(*)$. We call
  - $(\mathbf{v}, \lambda)$ an eigen-pair of $\mathbf{A}$
  - λ an eigenvalue of $\mathbf{A}$; $\mathbf{v}$ an eigenvector of $\mathbf{A}$ associated with λ
- if $(\mathbf{v}, \lambda)$ is an eigen-pair of $\mathbf{A}$, $(\alpha \mathbf{v}, \lambda)$ is also an eigen-pair for any $\alpha \in \mathbb{C}, \alpha \neq 0$
- unless specified, we will assume $\|\mathbf{v}\|_2 = 1$ in the sequel

Eigenvalues and Eigenvectors

Fact: Every $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) has n eigenvalues.

- from the eigenvalue problem we see that

$$\begin{aligned} \mathbf{A}\mathbf{v} = \lambda\mathbf{v} \text{ for some } \mathbf{v} \neq \mathbf{0} \quad &\Longleftrightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \text{ for some } \mathbf{v} \neq \mathbf{0} \\ &\Longleftrightarrow \quad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \end{aligned}$$

- let $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$, called the characteristic polynomial of $\mathbf{A}$
- from the determinant def., it can be shown that $p(\lambda)$ is a polynomial of degree n, viz., $p(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \ldots + \alpha_n\lambda^n$ where α_i's depend on $\mathbf{A}$
- as $p(\lambda)$ is a polynomial of degree n, it can be factored as $p(\lambda) = \prod_{i=1}^n (\lambda_i - \lambda)$ where $\lambda_1, \ldots, \lambda_n$ are the roots of $p(\lambda)$
- we have $\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Longleftrightarrow \lambda \in \{\lambda_1, \ldots, \lambda_n\}$

Eigenvalues and Eigenvectors

Let $\lambda_1, \ldots, \lambda_n$ denote the n eigenvalues of $\mathbf{A}$. We write

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i, \qquad i = 1, \ldots, n,$$

where $\mathbf{v}_i$ denotes an eigenvector of $\mathbf{A}$ associated with λ_i.

- we should be careful about the meaning of n eigenvalues: *they are defined as the n roots of the characteristic polynomial $p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I})$*
- example: consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

  - from the original definition $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$, one can verify that $\lambda = 1$ is the only eigenvalue of $\mathbf{A}$
  - from the characteristic polynomial, which is $p(\lambda) = (1 - \lambda)^2$, we see two roots $\lambda_1 = \lambda_2 = 1$ as two eigenvalues

Eigenvalues and Eigenvectors

Fact: an eigenvalue can be complex even if $\mathbf{A}$ is real.

- a polynomial $p(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \ldots + \alpha_n\lambda^n$ with real coefficients α_i's can have complex roots
- example: consider
$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$
  - we have $p(\lambda) = \lambda^2 + 1$, so $\lambda_1 = \boldsymbol{j}$, $\lambda_2 = -\boldsymbol{j}$

Fact: if $\mathbf{A}$ is real and there exists a real eigenvalue λ of $\mathbf{A}$, the associated eigenvector $\mathbf{v}$ can be taken as real.

- obviously, when $\mathbf{A} - \lambda\mathbf{I}$ is real we can define $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$ on $\mathbb{R}^n$
- or, if $\mathbf{v}$ is a complex eigenvector of a real $\mathbf{A}$ associated with a real λ, we can write $\mathbf{v} = \mathbf{v}_{\rm R} + \boldsymbol{j}\mathbf{v}_{\rm I}$, where $\mathbf{v}_{\rm R}, \mathbf{v}_{\rm I} \in \mathbb{R}^n$. It is easy to verify that $\mathbf{v}_{\rm R}$ and $\mathbf{v}_{\rm I}$ are eigenvectors associated with λ

Further Discussion: Repeated Eigenvalues

- w.l.o.g., order $\lambda_1, \ldots, \lambda_n$ such that $\{\lambda_1, \ldots \lambda_k\}$, $k \leq n$, is the set of all distinct eigenvalues of $\mathbf{A}$; i.e., $\lambda_i \neq \lambda_j$ for all $i, j \in \{1, \ldots, k\}$, $i \neq j$; $\lambda_i \in \{\lambda_1, \ldots \lambda_k\}$ for all $i \in \{1, \ldots, n\}$

- denote μ_i as the number of repeated eigenvalues of λ_i, $i = 1, \ldots, k$
  - μ_i is called the algebraic multiplicity of the eigenvalue λ_i

- every λ_i can have more than one eigenvector (scaling not counted)
  - if $\dim \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I}) = r$, we can find r linearly independent $\mathbf{v}_i$'s
  - denote $\gamma_i = \dim \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I})$, $i = 1, \ldots, k$
  - γ_i is called the geometric multiplicity of the eigenvalue λ_i

Property 3.1. We have $\mu_i \geq \gamma_i$ for all $i = 1, \ldots, k$ (not trivial, requires a proof)

- **Implication:** no. of repeated eigenvalues $\geq$ no. of linearly indep. eigenvectors

Eigendecomposition

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) is said to be diagonalizable, or admit an eigendecomposition, if there exists a nonsingular $\mathbf{V} \in \mathbb{C}^{n \times n}$ such that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1},$$

where $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$.

- in defining diagonalizability, we didn't say that $(\mathbf{v}_i, \lambda_i)$ has to be an eigen-pair of $\mathbf{A}$. But

$$\begin{aligned} \mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} &\iff \mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}, \ \mathbf{V} \text{ nonsingular} \\ &\iff \mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i, \ i = 1, \ldots, n, \ \mathbf{V} \text{ nonsingular} \end{aligned}$$

  Also, $\lambda_1, \ldots, \lambda_n$ must be the n eigenvalues of $\mathbf{A}$; this can be seen from the characteristic polynomial $\det(\mathbf{A} - \lambda \mathbf{I}) = \det(\mathbf{\Lambda} - \lambda \mathbf{I}) = \prod_{i=1}^n (\lambda_i - \lambda)$

- the non-trivial part lies in finding n linearly independent eigenvectors

Eigendecomposition

If $\mathbf{A}$ admits an eigendecomposition, the following properties can be shown (easily):

- $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$
- $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$
- the eigenvalues of $\mathbf{A}^k$ are $\lambda_1^k, \ldots, \lambda_n^k$
- $\mathrm{rank}(\mathbf{A})$ = number of nonzero eigenvalues of $\mathbf{A}$
- suppose that $\mathbf{A}$ is also nonsingular. Then, $\mathbf{A}^{-1} = \mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{V}^{-1}$

Note: the first three properties can be shown to be valid for any $\mathbf{A}$; the fourth property may not be valid when $\mathbf{A}$ does not admit an eigendecomposition

Eigendecomposition

Question: Does every $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) admit an eigendecomposition?

- the answer is no.
- counter example: consider

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  - the characteristic polynomial is $p(\lambda) = -\lambda^3$, so $\lambda_1 = \lambda_2 = \lambda_3 = 0$
  - it is easy to see that

$$\mathcal{N}(\mathbf{A} - \lambda_1 \mathbf{I}) = \mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^T)^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

  - any selection of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{N}(\mathbf{A})$ is linearly dependent

Eigendecomposition

Question: under which conditions can a matrix admit an eigendcomposition?

- there exist matrix subclasses in which eigendecomposition is guaranteed to exist
  - one example is the circulant matrix subclass, as seen in the last lecture
  - another example is the Hermitian matrix subclass, as we will see
- there exist simple sufficient conditions under which eigendec. exists

Eigendecomposition

Property 3.2. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$), and suppose that λ_i's are ordered such that $\{\lambda_1, \ldots, \lambda_k\}$ is the set of all distinct eigenvalues of $\mathbf{A}$. Also, let $\mathbf{v}_i$ be *any* eigenvector associated with λ_i. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must be linearly independent.

Implications:

- if all the eigenvalues of $\mathbf{A}$ are distinct, i.e.,

$$\lambda_i \neq \lambda_j, \quad \text{for all } i, j \in \{1, \ldots, n\} \text{ with } i \neq j,$$

  then $\mathbf{A}$ admits an eigendecomposition

  – to have all the eigenvalues to be distinct is not that hard, as we will see later

- $\mathbf{A}$ admits an eigendcomposition if and only if $\mu_i = \gamma_i$ for all i

Eigendecomposition for Hermitian & Real Symmetric Matrices

Consider the Hermitian matrix subclass.

Property 3.3. Let $\mathbf{A} \in \mathbb{H}^n$.

1. the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{A}$ are real
2. suppose that λ_i's are ordered such that $\{\lambda_1, \ldots, \lambda_k\}$ is the set of all distinct eigenvalues of $\mathbf{A}$. Also, let $\mathbf{v}_i$ be *any* eigenvector associated with λ_i. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must be orthonormal.

- the above results apply to real symmetric matrices; recall $\mathbf{A} \in \mathbb{S}^n \Longrightarrow \mathbf{A} \in \mathbb{H}^n$
- Corollary: for a real symmetric matrix, all eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ can be chosen as real

Eigendecomposition for Real Symmetric & Hermitian Matrices

Theorem 3.1. Every $\mathbf{A} \in \mathbb{H}^n$ admits an eigendecomposition

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H,$$

where $\mathbf{V} \in \mathbb{C}^{n \times n}$ is unitary; $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all i. Also, if $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{V}$ can be taken as real orthogonal.

- a consequence of a more powerful decomposition, namely, the Schur decomposition; we will go through it later
- does not require the assumption of distinct eigenvalues
- Corollary: if $\mathbf{A}$ is Hermitian or real symmetric, $\mu_i = \gamma_i$ for all i (no. of repeated eigenvalues = no. of linearly indep. eigenvectors)

Power Method

- a method of numerically computing an eigenvector of a given matrix
- simple
- not the best in convergence speed
  - a comprehensive coverage of various computational methods for the eigenvalue problem can be found in the textbook **[Golub-Van Loan'12]**
- suitable for large-scale sparse problems, e.g., PageRank

Power Method

- assumptions:
  - $\mathbf{A}$ admits an eigendecomposition
  - λ_i's are ordered such that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$
  - $|\lambda_1| > |\lambda_2|$
  - we have an initial guess $\mathbf{x}$ that satisfies $[\mathbf{V}^{-1}\mathbf{x}]_1 \neq 0$ (random guess should do)
- consider $\mathbf{A}^k\mathbf{x}$. Let $\boldsymbol{\alpha} = \mathbf{V}^{-1}\mathbf{x}$, and observe

$$\mathbf{A}^k\mathbf{x} = \mathbf{V}\boldsymbol{\Lambda}^k\mathbf{V}^{-1}\mathbf{x} = \sum_{i=1}^n \alpha_i \lambda_i^k \mathbf{v}_i = \alpha_1\lambda_1^k\Bigg(\mathbf{v}_1 + \underbrace{\sum_{i=2}^n \frac{\alpha_i}{\alpha_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k \mathbf{v}_i}_{=\mathbf{r}_k}\Bigg)$$

  where $\mathbf{r}_k$ is a residual and has

$$\|\mathbf{r}_k\|_2 \leq \sum_{i=2}^n \left|\frac{\alpha_i}{\alpha_1}\right| \left|\frac{\lambda_i}{\lambda_1}\right|^k \|\mathbf{v}_i\|_2 \leq \left|\frac{\lambda_2}{\lambda_1}\right|^k \sum_{i=2}^n \left|\frac{\alpha_i}{\alpha_1}\right|$$

- convergence: let $c_k = \frac{|\alpha_1||\lambda|^k}{\alpha_1 \lambda_1^k}$ (note $|c_k| = 1$). We have

$$\lim_{k\to\infty} c_k \frac{\mathbf{A}^k\mathbf{x}}{\|\mathbf{A}^k\mathbf{x}\|_2} = \mathbf{v}_1$$

Power Method

Algorithm: Power Method
input: $\mathbf{A} \in \mathbb{C}^{n \times n}$ and a starting point $\mathbf{v}^{(0)} \in \mathbb{C}^n$
$k = 0$
repeat
  $\tilde{\mathbf{v}}^{(k+1)} = \mathbf{A}\mathbf{v}^{(k)}$
  $\mathbf{v}^{(k+1)} = \tilde{\mathbf{v}}^{(k+1)} / \|\tilde{\mathbf{v}}^{(k+1)}\|_2$
  $k := k + 1$
until a stopping rule is satisfied
output: $\mathbf{v}^{(k)}$

- it can be verified that $\mathbf{v}^{(k)} = \dfrac{\mathbf{A}^k \mathbf{v}^{(0)}}{\|\mathbf{A}^k \mathbf{v}^{(0)}\|_2}$
- complexity per iteration: $\mathcal{O}(n^2)$, or $\mathcal{O}(\mathrm{nnz}(\mathbf{A}))$ for sparse $\mathbf{A}$
- convergence rate depends on $\left|\frac{\lambda_2}{\lambda_1}\right|$; slower if $|\lambda_2|$ is closer to $|\lambda_1|$

Deflation

- the power method finds the largest eigenvalue (in modulus) only
- how can we compute all the eigenvalues and eigenvectors?
- there are many ways and let's consider a simple method called deflation
- consider a Hermitian $\mathbf{A}$ with $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$, and note the outer-product representation

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^H.$$

- Deflation: use the power method to obtain $\mathbf{v}_1, \lambda_1$, do the subtraction

$$\mathbf{A} := \mathbf{A} - \lambda_1 \mathbf{v}_1 \mathbf{v}_1^H = \sum_{i=2}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^H,$$

  and repeat until all the eigenvectors and eigenvalues are found

  – if we want the first k eigen-pairs only, deflation can also do that

Schur Decomposition

Theorem 3.2. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$, and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. The matrix $\mathbf{A}$ admits a decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^H,$$

for some unitary $\mathbf{U} \in \mathbb{C}^{n\times n}$ and for some upper triangular $\mathbf{T} \in \mathbb{C}^{n\times n}$ with $t_{ii} = \lambda_i$ for all i. If $\mathbf{A}$ is real and $\lambda_1, \ldots, \lambda_n$ are all real, $\mathbf{U}$ and $\mathbf{T}$ can be taken as real.

- we will call the above decomposition the Schur decomposition in the sequel
- some insight: Suppose $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^H$ for some unitary $\mathbf{U}$ and upper triangular $\mathbf{T}$, but it's not known if $t_{ii} = \lambda_i$. Then

  $$\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{T} - \lambda\mathbf{I}) = \textstyle\prod_{i=1}^n (t_{ii} - \lambda)$$

  This implies that $t_{11}, \ldots, t_{nn}$ are the eigenvalues of $\mathbf{A}$
- see the accompanying note for the proof of Theorem 3.2

Schur Decomposition

- the Schur decomposition is a powerful tool
- e.g., we can use it to show that for *any* square $\mathbf{A}$ (with or without eigendec.),
  - $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$
  - $\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$
  - the eigenvalues of $\mathbf{A}^k$ are $\lambda_1^k, \ldots, \lambda_n^k$
- we may use it to prove the convergence of the power method when eigendecomposition does not exist
- the Jordan canonical form, which we will not teach, requires the Schur decomposition as the first key step

Implications of the Schur Decomposition

- proof of Theorem 3.1:
  - let $\mathbf{A}$ be Hermitian, and let $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^H$ be its Schur decomposition. Observe

$$\mathbf{0} = \mathbf{A} - \mathbf{A}^H = \mathbf{U}\mathbf{T}\mathbf{U}^H - \mathbf{U}\mathbf{T}^H\mathbf{U}^H = \mathbf{U}(\mathbf{T} - \mathbf{T}^H)\mathbf{U}^H \quad \Longleftrightarrow \quad \mathbf{0} = \mathbf{T} - \mathbf{T}^H$$

  - since $\mathbf{T}$ is upper triangular and $\mathbf{T}^H$ is lower triangular, $\mathbf{T} = \mathbf{T}^H$ implies that $\mathbf{T}$ is diagonal; thus, the Schur decomposition is also the eigendecomposition
  - similar results apply to real symmetric $\mathbf{A}$, except that we use real $\mathbf{T}, \mathbf{U}$
  - note: $\mathbf{T} = \mathbf{T}^H$ also implies that t_{ii}'s are real; so the proof also confirms that λ_i's are real

- skew-Hermitian matrices: $\mathbf{A} \in \mathbb{C}^{n \times n}$ is said to be skew-Hermitian if $\mathbf{A}^H = -\mathbf{A}$
  - by the Schur decomposition, we can show that any skew-Hermitian $\mathbf{A}$ admits an eigendecomposition with unitary $\mathbf{V}$ and the eigenvalues are purely imaginary

Implications of Schur Decomposition

- another result from the Schur decomposition:

  Proposition 3.1. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$. For every $\varepsilon > 0$, there exists a matrix $\tilde{\mathbf{A}} \in \mathbb{C}^{n \times n}$ such that the n eigenvalues of $\tilde{\mathbf{A}}$ are distinct and

$$\|\mathbf{A} - \tilde{\mathbf{A}}\|_F \leq \varepsilon.$$

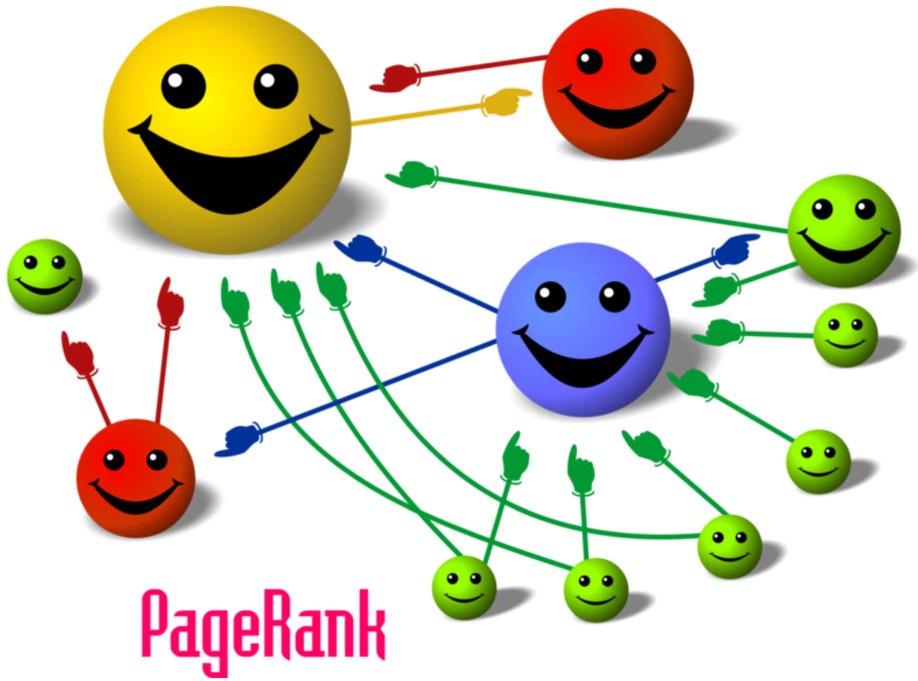
- Implication: for any square $\mathbf{A}$, we can always find an $\tilde{\mathbf{A}}$ that is arbitrarily close to $\mathbf{A}$ and admits an eigendecomposition

- proof:
  - let $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^H$ be the Schur dec. of $\mathbf{A}$
  - let $\tilde{\mathbf{A}} = \mathbf{U}(\mathbf{T}+\mathbf{D})\mathbf{U}^H$, where $\mathbf{D} = \mathrm{Diag}(d_1, \ldots, d_n)$ is such that $|d_i| \leq \left(\frac{\varepsilon}{n}\right)^{1/2}$ for all i, and $t_{11} + d_1, \ldots, t_{nn} + d_n$ are distinct
  - $\|\mathbf{A} - \tilde{\mathbf{A}}\|_F^2 = \|\mathbf{D}\|_F^2 \leq \varepsilon$, the eigenvalues of $\tilde{\mathbf{A}}$ are $t_{11} + d_1, \ldots, t_{nn} + d_n$
  - by Property 3.2, $\tilde{\mathbf{A}}$ admits an eigendecomposition

PageRank: A Case Study

- PageRank is an algorithm used by Google to rank the pages of a search result.

- the idea is to use counts of links of various pages to determine pages' importance.



Source: Wiki.

- further reading: **[Bryan-Tanya2006]**

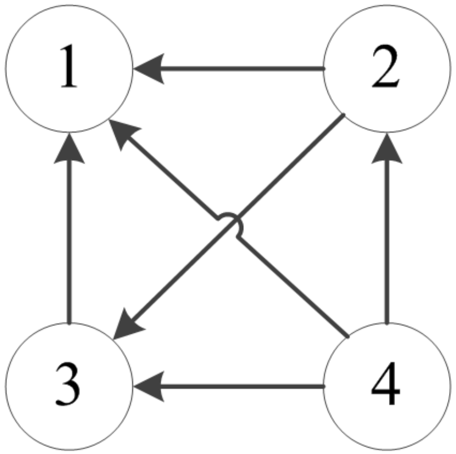
PageRank Model

- Model:

$$\sum_{j \in \mathcal{L}_i} \frac{v_j}{c_j} = v_i, \quad i = 1, \ldots, n,$$

  where c_j is the number of outgoing links from page j; $\mathcal{L}_i$ is the set of pages with a link to page i; v_i is the importance score of page i.

- example:



$$\overbrace{\begin{bmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}^{\mathbf{v}} = \overbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}^{\mathbf{v}}.$$

PageRank Problem

- let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix such that $a_{ij} = 1/c_j$ if $j \in \mathcal{L}_i$ and $a_{ij} = 0$ if $j \notin \mathcal{L}_i$
- Problem: find a non-negative $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \mathbf{v}$
  - $\mathbf{A}$ is extremely large and sparse, and we want to use the power method
- Questions:
  - does a solution to $\mathbf{A}\mathbf{v} = \mathbf{v}$ exist? Or, is $\lambda = 1$ an eigenvalue of $\mathbf{A}$?
  - does $\mathbf{A}\mathbf{v} = \mathbf{v}$ have a non-negative solution? Or, does a non-negative eigenvector associated with $\lambda = 1$ exist?
  - is the solution to $\mathbf{A}\mathbf{v} = \mathbf{v}$ unique? Or, would there exist more than one eigenvector associated with $\lambda = 1$?
    - a unique solution is desired for this problem
  - is $\lambda = 1$ the only eigenvalue that is the largest in modulus?
    - this is required for the power method

Some Notation and Conventions

- notation:
  - $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for all i
  - $\mathbf{x} > \mathbf{y}$ means that $x_i > y_i$ for all i
  - $\mathbf{x} \not\geq \mathbf{y}$ means that $\mathbf{x} \geq \mathbf{y}$ does not hold
  - the same notations apply to matrices

- conventions:
  - $\mathbf{x}$ is said to be non-negative if $\mathbf{x} \geq \mathbf{0}$, and non-positive if $-\mathbf{x} \geq \mathbf{0}$
  - $\mathbf{x}$ is said to be positive if $\mathbf{x} > \mathbf{0}$, and negative if $-\mathbf{x} > \mathbf{0}$
  - the same conventions apply to matrices
  - a square $\mathbf{A}$ is said to be column-stochastic if $\mathbf{A} \geq \mathbf{0}$ and $\mathbf{A}^T\mathbf{1} = \mathbf{1}$
    * a column-stochastic $\mathbf{A}$ has every column $\mathbf{a}_i$ satisfying $\mathbf{a}_i^T\mathbf{1} = \sum_{j=1}^n a_{ji} = 1$

PageRank Matrix Properties

- in PageRank, $\mathbf{A}$ is column-stochastic if all pages have outgoing links
  - see the literature to see how to deal with cases where some pages do not have outgoing links (dangling nodes)

Property 3.4. Let $\mathbf{A}$ be column-stochastic. Then,

1. $\lambda = 1$ is an eigenvalue of $\mathbf{A}$
2. $|\lambda| \leq 1$ for any eigenvalue λ of $\mathbf{A}$

- Implications:
  - a solution to $\mathbf{A}\mathbf{v} = \mathbf{v}$ does exist, though it doesn't say if $\mathbf{v} \geq \mathbf{0}$ or not
  - $\lambda = 1$ is an eigenvalue that has the largest modulus, but we don't know if it is the *only* eigenvalue that has the largest modulus

- we resort to non-negative matrix theory to answer the rest of the questions

Non-Negative Matrix Theory

Theorem 3.3 (Perron-Frobenius)**.** Let $\mathbf{A}$ be square positive. There exists an eigenvalue ρ of $\mathbf{A}$ such that

1. ρ is real and $\rho > 0$
2. $|\lambda| < \rho$ for any eigenvalue λ of $\mathbf{A}$ with $\lambda \neq \rho$
3. there exists a positive eigenvector associated with ρ
4. the algebraic multiplicity of ρ is 1 (so the geometric multiplicity of ρ is also 1)

A weaker result for general non-negative matrices:

Theorem 3.4. Let $\mathbf{A}$ be square non-negative. There exists an eigenvalue ρ of $\mathbf{A}$ such that

1. ρ is real and $\rho \geq 0$
2. $|\lambda| \leq \rho$ for any eigenvalue λ of $\mathbf{A}$
3. there exists a non-negative eigenvector associated with ρ

PageRank Matrix Properties

- further implication by Theorem 3.4:
  - a non-negative solution to $\mathbf{A}\mathbf{v} = \mathbf{v}$ exists, though it doesn't say if there exists another solution
  - even worse, it is not known if there exists another solution $\mathbf{v}$ such that $\mathbf{v} \not\geq \mathbf{0}$

PageRank Matrix Properties

- PageRank actually considers a modified version of $\mathbf{A}$

$$\tilde{\mathbf{A}} = (1-\beta)\mathbf{A} + \beta \begin{bmatrix} 1/n & \ldots & 1/n \\ \vdots & & \vdots \\ 1/n & \ldots & 1/n \end{bmatrix}$$

  where $0 < \beta < 1$ (typical value is $\beta = 0.15$)

- $\tilde{\mathbf{A}}$ is positive

- further implications by Theorem 3.3:
  - $\lambda = 1$ is the *only* eigenvalue that has the largest modulus
  - there exists *only* one eigenvector associated with $\lambda = 1$; that eigenvector is either positive or negative
  - so the power method should work

References

[Golub-Van Loan'12] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd edition, JHU Press, 2012.

[Bryan-Tanya2006] K. Bryan and L. Tanya, "The $25,000,000,000$ eigenvector: The linear algebra behind Google," *SIAM Review*, vol. 48, no. 3, pp. 569–581, 2006.