ENGG5781 Matrix Analysis and Computations
Lecture 3: Eigenvalues and Eigenvectors

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Lecture 3: Eigenvalues and Eigenvectors

• facts about eigenvalues and eigenvectors

• eigendecomposition, the case of Hermitian and real symmetric matrices

• power method

• Schur decomposition

• PageRank: a case study
Notation and Conventions

- a square matrix $\mathbf{A}$ is said to be symmetric if $a_{ij} = a_{ji}$ for all $i, j$ with $i \neq j$, or equivalently, if $\mathbf{A}^T = \mathbf{A}$

  - example:

    $$\mathbf{A} = \begin{bmatrix} 1 & -0.5 & 3 \\ -0.5 & -2 & 0.9 \\ 3 & 0.9 & 0.1 \end{bmatrix}$$

- a square matrix $\mathbf{A}$ is said to be Hermitian if $a_{ij} = a_{ji}^*$ for all $i, j$ with $i \neq j$, or equivalently, if $\mathbf{A}^H = \mathbf{A}$

- we denote the set of all $n \times n$ real symmetric matrices by $\mathbb{S}^n$

- we denote the set of all $n \times n$ complex Hermitian matrices by $\mathbb{H}^n$
Notation and Conventions

- note the following subtleties:
  - by definition, a real symmetric matrix is also Hermitian
  - when we say that a matrix is Hermitian, we often imply that the matrix may be complex (at least for this course); a real Hermitian matrix is simply real symmetric
  - we can have a complex symmetric matrix, though we will not study it
Main Results

A matrix $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) is said to admit an eigendecomposition if there exists a nonsingular $V \in \mathbb{C}^{n \times n}$ and a collection of scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$A = V \Lambda V^{-1},$$

where $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n)$.

- the above $(V, \Lambda)$ satisfies $A v_i = \lambda_i v_i$ for $i = 1, \ldots, n$, which are eigen-equations
- $v_1, \ldots, v_n$ are required to be linearly independent
- eigendecomposition does not always exist
Main Results

A real symmetric matrix $A \in S^n$ always admits an eigendecomposition

$$A = V \Lambda V^T$$

where $V \in \mathbb{R}^{n \times n}$ is orthogonal; $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all $i$.

A Hermitian matrix $A \in \mathbb{H}^n$ always admits an eigendecomposition

$$A = V \Lambda V^H$$

where $V \in \mathbb{C}^{n \times n}$ is unitary; $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all $i$.

- differences: a Hermitian or real symmetric matrix always has
  - an eigendecomposition
  - real $\lambda_i$'s
  - a $V$ that is not only nonsingular but also unitary
Eigenvalues and Eigenvectors

We start with the basic definition of eigenvalues and eigenvectors.

**Problem:** given a $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$), find a vector $v \in \mathbb{C}^n$ with $v \neq 0$ such that

$$Av = \lambda v,$$

for some $\lambda \in \mathbb{C}$ \hspace{1cm} (*)

- (*) is called an eigenvalue problem or eigen-equation
- let $(v, \lambda)$ be a solution to (*). We call
  - $(v, \lambda)$ an eigen-pair of $A$
  - $\lambda$ an eigenvalue of $A$; $v$ an eigenvector of $A$ associated with $\lambda$
- if $(v, \lambda)$ is an eigen-pair of $A$, $(\alpha v, \lambda)$ is also an eigen-pair for any $\alpha \in \mathbb{C}, \alpha \neq 0$
- unless specified, we will assume $\|v\|_2 = 1$ in the sequel
**Eigenvalues and Eigenvectors**

**Fact:** Every $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) has $n$ eigenvalues.

- from the eigenvalue problem we see that
  
  $$Av = \lambda v \text{ for some } v \neq 0 \iff (A - \lambda I)v = 0 \text{ for some } v \neq 0$$
  
  $$\iff \det(A - \lambda I) = 0$$

- let $p(\lambda) = \det(A - \lambda I)$, called the characteristic polynomial of $A$

- from the determinant def., it can be shown that $p(\lambda)$ is a polynomial of degree $n$, viz., $p(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \ldots + \alpha_n \lambda^n$ where $\alpha_i$’s depend on $A$

- as $p(\lambda)$ is a polynomial of degree $n$, it can be factored as $p(\lambda) = \prod_{i=1}^{n}(\lambda_i - \lambda)$ where $\lambda_1, \ldots, \lambda_n$ are the roots of $p(\lambda)$

- we have $\det(A - \lambda I) = 0 \iff \lambda \in \{\lambda_1, \ldots, \lambda_n\}$
Eigenvalues and Eigenvectors

Let $\lambda_1, \ldots, \lambda_n$ denote the $n$ eigenvalues of $A$. We write

$$A v_i = \lambda_i v_i, \quad i = 1, \ldots, n,$$

where $v_i$ denotes an eigenvector of $A$ associated with $\lambda_i$.

- we should be careful about the meaning of $n$ eigenvalues: they are defined as the $n$ roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$

- example: consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- from the original definition $A v = \lambda v$, one can verify that $\lambda = 1$ is the only eigenvalue of $A$

- from the characteristic polynomial, which is $p(\lambda) = (1 - \lambda)^2$, we see two roots $\lambda_1 = \lambda_2 = 1$ as two eigenvalues
Eigenvalues and Eigenvectors

Fact: an eigenvalue can be complex even if $A$ is real.

- a polynomial $p(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \ldots + \alpha_n \lambda^n$ with real coefficients $\alpha_i$'s can have complex roots

- example: consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
  - we have $p(\lambda) = \lambda^2 + 1$, so $\lambda_1 = j$, $\lambda_2 = -j$

Fact: if $A$ is real and there exists a real eigenvalue $\lambda$ of $A$, the associated eigenvector $v$ can be taken as real.

- obviously, when $A - \lambda I$ is real we can define $N(A - \lambda I)$ on $\mathbb{R}^n$

- or, if $v$ is a complex eigenvector of a real $A$ associated with a real $\lambda$, we can write $v = v_R + jv_I$, where $v_R, v_I \in \mathbb{R}^n$. It is easy to verify that $v_R$ and $v_I$ are eigenvectors associated with $\lambda$
Further Discussion: Repeated Eigenvalues

• w.l.o.g., order \( \lambda_1, \ldots, \lambda_n \) such that \( \{\lambda_1, \ldots, \lambda_k\}, \ k \leq n, \) is the set of all distinct eigenvalues of \( A; \) i.e., \( \lambda_i \neq \lambda_j \) for all \( i, j \in \{1, \ldots, k\}, \ i \neq j; \lambda_i \in \{\lambda_1, \ldots, \lambda_k\} \) for all \( i \in \{1, \ldots, n\} \)

• denote \( \mu_i \) as the number of repeated eigenvalues of \( \lambda_i, \ i = 1, \ldots, k \)
  – \( \mu_i \) is called the algebraic multiplicity of the eigenvalue \( \lambda_i \)

• every \( \lambda_i \) can have more than one eigenvector (scaling not counted)
  – if \( \dim \mathcal{N}(A - \lambda_i \mathbf{I}) = r \), we can find \( r \) linearly independent \( v_i \)'s
  – denote \( \gamma_i = \dim \mathcal{N}(A - \lambda_i \mathbf{I}), \ i = 1, \ldots, k \)
  – \( \gamma_i \) is called the geometric multiplicity of the eigenvalue \( \lambda_i \)

Property 3.1. We have \( \mu_i \geq \gamma_i \) for all \( i = 1, \ldots, k \) (not trivial, requires a proof)
  – Implication: no. of repeated eigenvalues \( \geq \) no. of linearly indep. eigenvectors
Eigendecomposition

A matrix $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) is said to be \textit{diagonalizable}, or admit an \textit{eigendecomposition}, if there exists a nonsingular $V \in \mathbb{C}^{n \times n}$ such that

$$A = V \Lambda V^{-1},$$

where $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n)$.

- in defining diagonalizability, we didn’t say that $(v_i, \lambda_i)$ has to be an eigen-pair of $A$. But

$$A = V \Lambda V^{-1} \iff AV = V \Lambda, \ V \text{ nonsingular}$$

$$\iff Av_i = \lambda_i v_i, \ i = 1, \ldots, n, \ V \text{ nonsingular}$$

Also, $\lambda_1, \ldots, \lambda_n$ must be the $n$ eigenvalues of $A$; this can be seen from the characteristic polynomial $\det(A - \lambda I) = \det(\Lambda - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda)$

- the non-trivial part lies in finding $n$ linearly independent eigenvectors
Eigendecomposition

If $A$ admits an eigendecomposition, the following properties can be shown (easily):

- $\det(A) = \prod_{i=1}^{n} \lambda_i$
- $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$
- The eigenvalues of $A^k$ are $\lambda_1^k, \ldots, \lambda_n^k$
- $\text{rank}(A) =$ number of nonzero eigenvalues of $A$
- Suppose that $A$ is also nonsingular. Then, $A^{-1} = V\Lambda^{-1}V^{-1}$

Note: the first three properties can be shown to be valid for any $A$; the fourth property may not be valid when $A$ does not admit an eigendecomposition
Eigendecomposition

Question: Does every $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) admit an eigendecomposition?

• the answer is no.

• counter example: consider

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

– the characteristic polynomial is $p(\lambda) = -\lambda^3$, so $\lambda_1 = \lambda_2 = \lambda_3 = 0$

– it is easy to see that

$$N(A - \lambda_1 I) = N(A) = \mathcal{R}(A^T)^\perp = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

– any selection of $v_1, v_2, v_3 \in N(A)$ is linearly dependent
Eigendecomposition

Question: under which conditions can a matrix admit an eigendecomposition?

• there exist matrix subclasses in which eigendecomposition is guaranteed to exist
  – one example is the circulant matrix subclass, as seen in the last lecture
  – another example is the Hermitian matrix subclass, as we will see

• there exist simple sufficient conditions under which eigendec. exists
Eigendecomposition

Property 3.2. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$), and suppose that $\lambda_i$’s are ordered such that $\{\lambda_1, \ldots, \lambda_k\}$ is the set of all distinct eigenvalues of $\mathbf{A}$. Also, let $\mathbf{v}_i$ be any eigenvector associated with $\lambda_i$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must be linearly independent.

Implications:

• if all the eigenvalues of $\mathbf{A}$ are distinct, i.e.,

$$\lambda_i \neq \lambda_j, \quad \text{for all } i, j \in \{1, \ldots, n\} \text{ with } i \neq j,$$

then $\mathbf{A}$ admits an eigendecomposition

– to have all the eigenvalues to be distinct is not that hard, as we will see later

• $\mathbf{A}$ admits an eigendecomposition if and only if $\mu_i = \gamma_i$ for all $i$
Eigendecomposition for Hermitian & Real Symmetric Matrices

Consider the Hermitian matrix subclass.

**Property 3.3.** Let $A \in \mathbb{H}^n$.

1. the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ are real

2. suppose that $\lambda_i$’s are ordered such that $\{\lambda_1, \ldots, \lambda_k\}$ is the set of all distinct eigenvalues of $A$. Also, let $v_i$ be any eigenvector associated with $\lambda_i$. Then $v_1, \ldots, v_k$ must be orthonormal.

- the above results apply to real symmetric matrices; recall $A \in S^n \implies A \in \mathbb{H}^n$

- **Corollary:** for a real symmetric matrix, all eigenvectors $v_1, \ldots, v_n$ can be chosen as real
Theorem 3.1. Every $A \in \mathbb{H}^n$ admits an eigendecomposition

$$A = V \Lambda V^H,$$

where $V \in \mathbb{C}^{n \times n}$ is unitary; $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all $i$. Also, if $A \in \mathbb{S}^n$, $V$ can be taken as real orthogonal.

- a consequence of a more powerful decomposition, namely, the Schur decomposition; we will go through it later

- does not require the assumption of distinct eigenvalues

- Corollary: if $A$ is Hermitian or real symmetric, $\mu_i = \gamma_i$ for all $i$ (no. of repeated eigenvalues $=$ no. of linearly indep. eigenvectors)
Power Method

- a method of numerically computing an eigenvector of a given matrix
- simple
- not the best in convergence speed
  - a comprehensive coverage of various computational methods for the eigenvalue problem can be found in the textbook [*Golub-Van Loan’12*]
- suitable for large-scale sparse problems, e.g., PageRank
Power Method

• assumptions:
  – $A$ admits an eigendecomposition
  – $\lambda_i$’s are ordered such that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$
  – $|\lambda_1| > |\lambda_2|$
  – we have an initial guess $x$ that satisfies $[V^{-1}x]_1 \neq 0$ (random guess should do)

• consider $A^kx$. Let $\alpha = V^{-1}x$, and observe

$$A^kx = V\Lambda^kV^{-1}x = \sum_{i=1}^{n} \alpha_i \lambda_i^k v_i = \alpha_1 \lambda_1^k \left( v_1 + \sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} \left( \frac{\lambda_i}{\lambda_1} \right)^k v_i \right)$$

where $r_k$ is a residual and has

$$\|r_k\|_2 \leq \sum_{i=2}^{n} \left| \frac{\alpha_i}{\alpha_1} \right| \left| \frac{\lambda_i}{\lambda_1} \right|^k \|v_i\|_2 \leq \left| \frac{\lambda_2}{\lambda_1} \right|^k \sum_{i=2}^{n} \left| \frac{\alpha_i}{\alpha_1} \right|$$

• convergence: let $c_k = \frac{|\alpha_1| |\lambda|^k}{\alpha_1 \lambda_1^k}$ (note $|c_k| = 1$). We have

$$\lim_{k \to \infty} c_k \frac{A^kx}{\|A^kx\|_2} = v_1$$
Power Method

Algorithm: Power Method
input: $A \in \mathbb{C}^{n \times n}$ and a starting point $v^{(0)} \in \mathbb{C}^n$

$k = 0$

repeat
\[
\tilde{v}^{(k+1)} = Av^{(k)}
\]
\[
v^{(k+1)} = \frac{\tilde{v}^{(k+1)}}{\|\tilde{v}^{(k+1)}\|_2}
\]
\[k := k + 1\]
until a stopping rule is satisfied
output: $v^{(k)}$

• it can be verified that $v^{(k)} = \frac{A^k v^{(0)}}{\|A^k v^{(0)}\|_2}$

• complexity per iteration: $O(n^2)$, or $O(\text{nnz}(A))$ for sparse $A$

• convergence rate depends on $\left|\frac{\lambda_2}{\lambda_1}\right|$; slower if $|\lambda_2|$ is closer to $|\lambda_1|$
Deflation

- the power method finds the largest eigenvalue (in modulus) only
- how can we compute all the eigenvalues and eigenvectors?
- there are many ways and let’s consider a simple method called deflation
- consider a Hermitian $A$ with $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$, and note the outer-product representation

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^H.$$  

- **Deflation**: use the power method to obtain $v_1, \lambda_1$, do the subtraction

$$A := A - \lambda_1 v_1 v_1^H = \sum_{i=2}^{n} \lambda_i v_i v_i^H,$$

and repeat until all the eigenvectors and eigenvalues are found

- if we want the first $k$ eigen-pairs only, deflation can also do that
Schur Decomposition

Theorem 3.2. Let $A \in \mathbb{C}^{n \times n}$, and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. The matrix $A$ admits a decomposition

$$A = UTU^H,$$

for some unitary $U \in \mathbb{C}^{n \times n}$ and for some upper triangular $T \in \mathbb{C}^{n \times n}$ with $t_{ii} = \lambda_i$ for all $i$. If $A$ is real and $\lambda_1, \ldots, \lambda_n$ are all real, $U$ and $T$ can be taken as real.

- we will call the above decomposition the Schur decomposition in the sequel
- some insight: Suppose $A$ can be written as $A = UTU^H$ for some unitary $U$ and upper triangular $T$, but it’s not known if $t_{ii} = \lambda_i$. Then

$$\det(A - \lambda I) = \det(T - \lambda I) = \prod_{i=1}^{n} (t_{ii} - \lambda)$$

This implies that $t_{11}, \ldots, t_{nn}$ are the eigenvalues of $A$
- see the accompanying note for the proof of Theorem 3.2
Schur Decomposition

• the Schur decomposition is a powerful tool

• e.g., we can use it to show that for any square $A$ (with or without eigendec.),
  
  $\det(A) = \prod_{i=1}^{n} \lambda_i$
  
  $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$
  
  the eigenvalues of $A^k$ are $\lambda_1^k, \ldots, \lambda_n^k$

• we may use it to prove the convergence of the power method when eigendecomposition does not exist

• the Jordan canonical form, which we will not teach, requires the Schur decomposition as the first key step
Implications of the Schur Decomposition

• proof of Theorem 3.1:
  – let $A$ be Hermitian, and let $A = UTU^H$ be its Schur decomposition. Observe

  $$0 = A - A^H = UTU^H - UT^H U^H = U(T - T^H)U^H \iff 0 = T - T^H$$

  – since $T$ is upper triangular and $T^H$ is lower triangular, $T = T^H$ implies that $T$ is diagonal; thus, the Schur decomposition is also the eigendecomposition

  – similar results apply to real symmetric $A$, except that we use real $T, U$

  – note: $T = T^H$ also implies that $t_{ii}$'s are real; so the proof also confirms that $\lambda_i$'s are real

• skew-Hermitian matrices: $A \in \mathbb{C}^{n \times n}$ is said to be skew-Hermitian if $A^H = -A$

  – by the Schur decomposition, we can show that any skew-Hermitian $A$ admits an eigendecomposition with unitary $V$ and the eigenvalues are purely imaginary
Implications of Schur Decomposition

• another result from the Schur decomposition:

**Proposition 3.1.** Let $A \in \mathbb{C}^{n \times n}$. For every $\varepsilon > 0$, there exists a matrix $\tilde{A} \in \mathbb{C}^{n \times n}$ such that the $n$ eigenvalues of $\tilde{A}$ are distinct and

$$\|A - \tilde{A}\|_F \leq \varepsilon.$$

• Implication: for any square $A$, we can always find an $\tilde{A}$ that is arbitrarily close to $A$ and admits an eigendecomposition

• proof:

  – let $D = \text{Diag}(d_1, \ldots, d_n)$ where $d_1, \ldots, d_n$ are chosen such that $|d_i| \leq \left(\frac{\varepsilon}{n}\right)^{1/2}$ for all $i$ and such that $t_{11} + d_1, \ldots, t_{nn} + d_n$ are distinct
  
  – let $\tilde{A} = UTU^H$ be the Schur dec. of $A$, and let $\tilde{A} = U(T + D)U^H$
  
  – we have $\|A - \tilde{A}\|_F^2 = \|D\|_F^2 \leq \varepsilon$
PageRank: A Case Study

- PageRank is an algorithm used by Google to rank the pages of a search result.
- The idea is to use counts of links of various pages to determine pages’ importance.

Source: Wiki.

- Further reading: [Bryan-Tanya2006]
PageRank Model

- Model:

\[ \sum_{j \in L_i} \frac{v_j}{c_j} = v_i, \quad i = 1, \ldots, n, \]

where \( c_j \) is the number of outgoing links from page \( j \); \( L_i \) is the set of pages with a link to page \( i \); \( v_i \) is the importance score of page \( i \).

- example:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{bmatrix}.
\]
PageRank Problem

- let $A \in \mathbb{R}^{n \times n}$ be a matrix such that $a_{ij} = 1/c_j$ if $j \in L_i$ and $a_{ij} = 0$ if $j \notin L_i$

- Problem: find a non-negative $v$ such that $Av = v$
  - $A$ is extremely large and sparse, and we want to use the power method

- Questions:
  - does a solution to $Av = v$ exist? Or, is $\lambda = 1$ an eigenvalue of $A$?
  - does $Av = v$ have a non-negative solution? Or, does a non-negative eigenvector associated with $\lambda = 1$ exist?
  - is the solution to $Av = v$ unique? Or, would there exist more than one eigenvector associated with $\lambda = 1$?
    * a unique solution is desired for this problem
  - is $\lambda = 1$ the only eigenvalue that is the largest in modulus?
    * this is required for the power method
Some Notation and Conventions

• notation:
  – $x \geq y$ means that $x_i \geq y_i$ for all $i$
  – $x > y$ means that $x_i > y_i$ for all $i$
  – $x \not\geq y$ means that $x \geq y$ does not hold
  – the same notations apply to matrices

• conventions:
  – $x$ is said to be non-negative if $x \geq 0$, and non-positive if $-x \geq 0$
  – $x$ is said to be positive if $x > 0$, and negative if $-x > 0$
  – the same conventions apply to matrices
  – a square $A$ is said to be column-stochastic if $A \geq 0$ and $A^T 1 = 1$
    * a column-stochastic $A$ has every column $a_i$ satisfying $a_i^T 1 = \sum_{j=1}^n a_{ji} = 1$
PageRank Matrix Properties

• in PageRank, $A$ is column-stochastic if all pages have outgoing links
  – see the literature to see how to deal with cases where some pages do not have outgoing links (dangling nodes)

Property 3.4. Let $A$ be column-stochastic. Then,
1. $\lambda = 1$ is an eigenvalue of $A$
2. $|\lambda| \leq 1$ for any eigenvalue $\lambda$ of $A$

• Implications:
  – a solution to $Av = v$ does exist, though it doesn’t say if $v \geq 0$ or not
  – $\lambda = 1$ is an eigenvalue that has the largest modulus, but we don’t know if it is the only eigenvalue that has the largest modulus

• we resort to non-negative matrix theory to answer the rest of the questions
Non-Negative Matrix Theory

**Theorem 3.3** (Perron-Frobenius). Let \( A \) be square positive. There exists an eigenvalue \( \rho \) of \( A \) such that

1. \( \rho \) is real and \( \rho > 0 \)
2. \( |\lambda| < \rho \) for any eigenvalue \( \lambda \) of \( A \) with \( \lambda \neq \rho \)
3. there exists a positive eigenvector associated with \( \rho \)
4. the algebraic multiplicity of \( \rho \) is 1 (so the geometric multiplicity of \( \rho \) is also 1)

A weaker result for general non-negative matrices:

**Theorem 3.4.** Let \( A \) be square non-negative. There exists an eigenvalue \( \rho \) of \( A \) such that

1. \( \rho \) is real and \( \rho \geq 0 \)
2. \( |\lambda| \leq \rho \) for any eigenvalue \( \lambda \) of \( A \)
3. there exists a non-negative eigenvector associated with \( \rho \)
PageRank Matrix Properties

- further implication by Theorem 3.4:
  - a non-negative solution to $A\mathbf{v} = \mathbf{v}$ exists, though it doesn’t say if there exists another solution
  - even worse, it is not known if there exists another solution $\mathbf{v}$ such that $\mathbf{v} \neq \mathbf{0}$
PageRank Matrix Properties

- PageRank actually considers a modified version of $A$

$$\tilde{A} = (1 - \beta)A + \beta \begin{bmatrix} 1/n & \ldots & 1/n \\ \vdots & \ddots & \vdots \\ 1/n & \ldots & 1/n \end{bmatrix}$$

where $0 < \beta < 1$ (typical value is $\beta = 0.15$)

- $\tilde{A}$ is positive

- further implications by Theorem 3.3:
  - $\lambda = 1$ is the only eigenvalue that has the largest modulus
  - there exists only one eigenvector associated with $\lambda = 1$; that eigenvector is either positive or negative
  - so the power method should work
References
