Lecture 2: Linear Representations and Least Squares

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1 Proof of Linear Independence of Vandemonde Matrices

Let k be any positive integer, and consider the following matrix

$$\mathbf{B} = \begin{bmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{k-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{k-1} \\ \vdots & & & & \vdots \\ 1 & z_k & z_k^2 & \dots & z_k^{k-1} \end{bmatrix} \in \mathbb{C}^{k \times k},$$

with $z_1, \dots, z_k \in \mathbb{C}$. We will show that $\mathbf{B}$ is nonsingular if z_i's are distinct. For now, let us assume this to be true and focus on showing the linear independence of $\mathbf{A}$. If $m \geq n$, we can represent $\mathbf{A}$ by

$$\mathbf{A}^T = [\ \mathbf{B} \ \ \times \ ]$$

with $k = n$; here, "$\times$" means parts that do not matter. By the rank definition, we have $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) \geq \text{rank}(\mathbf{B}) = n$. Since we also have $\text{rank}(\mathbf{A}) \leq n$, we obtain the result $\text{rank}(\mathbf{A}) = n$. Moreover, if $m \leq n$ we can represent $\mathbf{A}$ by

$$\mathbf{A} = [\ \mathbf{B}^T \ \ \times \ ]$$

with $k = m$. Following the same argument as above, we obtain $\text{rank}(\mathbf{A}) = m$. Thus we have established the result that $\mathbf{A}$ has full rank.

Now, we show that $\mathbf{B}$ is nonsingular if z_i's are distinct. Observe that

$$\mathbf{B}\boldsymbol{\alpha} = \mathbf{0} \quad \Longleftrightarrow \quad p(z_i) = 0, \ i = 1, \dots, k \tag{1}$$

where

$$p(z) = \alpha_1 + \alpha_2 z + \alpha_3 z^2 + \dots + \alpha_k z^{k-1}$$

denotes a polynomial of degree $k-1$. On one hand, the condition on the R.H.S. of (1) implies that $z_1, \dots, z_k$ are the roots of $p(z)$. On the other hand, it is known that a polynomial of degree $k-1$ has $k-1$ roots, and no more. Consequently, the above two statements contradict to each other if we have $z_i \neq z_j$ for all i, j with $i \neq j$. Hence, we have shown that $\mathbf{B}$ must be nonsingular if z_i's are distinct.