# **ENGG 5781** Matrix Analysis and Computations Lecture 1: Basic Concepts

Wing-Kin (Ken) Ma

2024–25 First Term

Department of Electronic Engineering The Chinese University of Hong Kong

# **Lecture 1: Basic Concepts**

- notation and conventions
- subspace, linear independence, basis, dimension
- rank, determinant, invertible matrices
- vector norms, inner product
- projections onto subspaces, orthogonal complements
- orthonormal basis, Gram Schmidt
- matrix multiplications and representations, block matrix manipulations
- complexity, floating point operations (flops)

$\mathbb{R}$	the set of real numbers, or real space
$\mathbb{C}$	the set of complex numbers, or complex space
$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	<i>n</i> -dimensional complex space
$\mathbb{R}^{m  imes n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m  imes n}$	set of all $m  imes n$ complex-valued matrices
X	column vector
$x_i, [\mathbf{x}]_i$	$i$ th entry of ${f x}$
$\mathbf{A}$	matrix
$a_{ij}, [\mathbf{A}]_{ij}$	$(i,j)$ th entry of ${f A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices; i.e, $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$
	for all $i, j$
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices; i.e, $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}^*$
	for all $i, j$

• vector:  $\mathbf{x} \in \mathbb{R}^n$  means that  $\mathbf{x}$  is a real-valued *n*-dimensional column vector; i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x_i \in \mathbb{R} \text{ for all } i.$$

Similarly,  $\mathbf{x} \in \mathbb{C}^n$  means that  $\mathbf{x}$  is a complex-valued *n*-dimensional column vector.

• transpose: let  $\mathbf{x} \in \mathbb{R}^n$ . The notation  $\mathbf{x}^T$  means that

 $\mathbf{x}^T = \begin{bmatrix} x_1, & x_2, & \dots, & x_n \end{bmatrix}.$ 

• Hermitian transpose: let  $\mathbf{x} \in \mathbb{C}^n$ . The notation  $\mathbf{x}^H$  means that

$$\mathbf{x}^H = \begin{bmatrix} x_1^*, & x_2^*, & \dots, & x_n^* \end{bmatrix},$$

where the superscript \* denotes the complex conjugate.

• matrix:  $\mathbf{A} \in \mathbb{R}^{m \times n}$  means that  $\mathbf{A}$  is real-valued  $m \times n$  matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad a_{ij} \in \mathbb{R} \text{ for all } i, j$$

Similarly,  $\mathbf{A} \in \mathbb{C}^{m \times n}$  means that  $\mathbf{A}$  is a complex-valued  $m \times n$  matrix.

• unless specified, we denote the *i*th column of a matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  as  $\mathbf{a}_i \in \mathbb{R}^m$ ; i.e.,

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1, & \mathbf{a}_2, & \dots, & \mathbf{a}_n \end{bmatrix}.$$

The same notation applies to  $\mathbf{A} \in \mathbb{C}^{m \times n}$ .

• transpose: let  $\mathbf{A} \in \mathbb{R}^{m \times n}$ . The notation  $\mathbf{A}^T$  means that

$$\mathbf{A}^{T} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}.$$

- or, we have  $\mathbf{B} = \mathbf{A}^T \iff b_{ij} = a_{ji}$  for all i, j.
- properties:

\* 
$$(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T$$
  
\*  $(\mathbf{A}^T)^T = \mathbf{A}$   
\*  $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ 

• Hermitian transpose: let  $\mathbf{A} \in \mathbb{C}^{m \times n}$ . The notation  $\mathbf{A}^H$  means that

$$\mathbf{A}^{H} = \begin{bmatrix} a_{11}^{*} & a_{21}^{*} & \dots & a_{m1}^{*} \\ a_{12}^{*} & a_{22}^{*} & \dots & a_{m2}^{*} \\ \vdots & & & \vdots \\ a_{1n}^{*} & a_{m2}^{*} & \dots & a_{mn}^{*} \end{bmatrix} \in \mathbb{C}^{n \times m}.$$

- or, we have 
$$\mathbf{B} = \mathbf{A}^H \iff b_{ij} = a_{ji}^*$$
 for all  $i, j$ .

- properties (same as transpose):

\* 
$$(\mathbf{AB})^{H} = \mathbf{B}^{H}\mathbf{A}^{H}$$
  
\*  $(\mathbf{A}^{H})^{H} = \mathbf{A}$   
\*  $(\mathbf{A} + \mathbf{B})^{H} = \mathbf{A}^{H} + \mathbf{B}^{H}$ 

• trace: let  $\mathbf{A} \in \mathbb{R}^{n \times n}$ . The trace of  $\mathbf{A}$  is

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

- properties:

\* 
$$\operatorname{tr}(\mathbf{A}^T) = \operatorname{tr}(\mathbf{A})$$
  
\*  $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$   
\*  $\operatorname{tr}(\mathbf{A}\mathbf{B}) = \operatorname{tr}(\mathbf{B}\mathbf{A})$  for  $\mathbf{A}, \mathbf{B}$  of appropriate sizes

• matrix power: let  $\mathbf{A} \in \mathbb{R}^{n \times n}$ . The notation  $\mathbf{A}^2$  means  $\mathbf{A}^2 = \mathbf{A}\mathbf{A}$ , and  $\mathbf{A}^k$  means

$$\mathbf{A}^k = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{k \mathbf{A}'\mathsf{s}}.$$

• all-one vectors: we use the notation

$$\mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

to denote a vector of all 1's.

- zero vectors or matrices: we use the notation **0** to denote either a vector of all zeros, or a matrix of all zeros.
- unit vectors: unit vectors are vectors that have only one nonzero element and the nonzero element is 1. We use the notation

$$\mathbf{e}_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T$$

to denote a unit vector with the nonzero element at the ith entry.

• identity matrix:

$$\mathbf{I} = egin{bmatrix} 1 & & & \ & 1 & & \ & & \ddots & \ & & & 1 \end{bmatrix},$$

where, as a convention, the empty entries are assumed to be zero.

• diagonal matrices: we use the notation

$$\operatorname{Diag}(a_1,\ldots,a_n) = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$$

to denote a diagonal matrix with diagonals  $a_1, \ldots, a_n$ . We also use the shorthand notation  $\text{Diag}(\mathbf{a}) = \text{Diag}(a_1, \ldots, a_n)$ .

- A matrix  $\mathbf{A} \in \mathbb{R}^{m imes n}$  is said to be
  - square if m = n;
  - tall if m > n;
  - fat if m < n.
- A matrix  $\mathbf{A} \in \mathbb{R}^{n imes n}$  is said to be
  - upper triangular if  $a_{ij} = 0$  for all i > j;
  - lower triangular if  $a_{ij} = 0$  for all i < j.

Examples:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 2 & 0 \\ \frac{1}{8} & 3 & 0 \end{bmatrix}.$$

### **Subspace**

A subset  $\mathcal{S}$  of  $\mathbb{R}^m$  is said to be a subspace if

$$\begin{array}{ll} \mathbf{x}, \mathbf{y} \in \mathcal{S}, \\ \alpha, \beta \in \mathbb{R} \end{array} \implies \alpha \mathbf{x} + \beta \mathbf{y} \in \mathcal{S}. \end{array}$$

- if S is a subspace and  $\mathbf{a}_1, \ldots, \mathbf{a}_n \in S$ , any linear combination of  $\mathbf{a}_1, \ldots, \mathbf{a}_n$ , i.e.,  $\sum_{i=1}^n \alpha_i \mathbf{a}_i$  for some  $\boldsymbol{\alpha} \in \mathbb{R}^n$ , lies in S.
- some quick facts: let  $\mathcal{S}_1, \mathcal{S}_2$  be subspaces of  $\mathbb{R}^m$ .
  - $S_1 + S_2$  is a subspace <sup>1</sup>
  - $\mathcal{S}_1 \cap \mathcal{S}_2$  is a subspace

<sup>1</sup>note the notation  $\mathcal{X} + \mathcal{Y} = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}\}.$ 

# Span

The span of a collection of vectors  $\mathbf{a}_1,\ldots,\mathbf{a}_n\in\mathbb{R}^m$  is defined as

span{
$$\mathbf{a}_1,\ldots,\mathbf{a}_n$$
} =  $\left\{ \mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{a}_i, \ \boldsymbol{\alpha} \in \mathbb{R}^n \right\}$ .

- the set of all linear combinations of  $\mathbf{a}_1,\ldots,\mathbf{a}_n$
- a subspace
- Question: any span is a subspace. But can any subspace be written as a span?

**Theorem 1.1.** Let S be a subspace of  $\mathbb{R}^m$ . There exists a positive integer n and a collection of vectors  $\mathbf{a}_1, \ldots, \mathbf{a}_n \in S$  such that  $S = \operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ .

- Implication: we can always represent a subspace by a span

# **Range Space and Nullspace**

The range space of  $\mathbf{A} \in \mathbb{R}^{m imes n}$  is defined as

$$\mathcal{R}(\mathbf{A}) = \{ \mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = \mathbf{A}\mathbf{x}, \ \mathbf{x} \in \mathbb{R}^n \}.$$

• essentially the same as span

The nullspace of  $\mathbf{A} \in \mathbb{R}^{m \times n}$  is defined as

$$\mathcal{N}(\mathbf{A}) = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0} \}.$$

- a nullspace is a subspace (verify as a mini exercise)
- by Theorem 1.1, we can represent a nullspace by  $\mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{B})$  for some  $\mathbf{B} \in \mathbb{R}^{n \times r}$  and positive integer r.

A collection of vectors  $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$  is said to be linearly independent if

$$\sum_{i=1}^{n} \alpha_i \mathbf{a}_i \neq \mathbf{0}, \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{R}^n \text{ with } \boldsymbol{\alpha} \neq \mathbf{0};$$

and linearly dependent otherwise.

• an equivalent way of defining linear dependence:  $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{R}^m$  is a linearly dependent vector set if there exists  $\boldsymbol{\alpha} \in \mathbb{R}^n$ ,  $\boldsymbol{\alpha} \neq \mathbf{0}$ , such that

$$\sum_{i=1}^n \alpha_i \mathbf{a}_i = \mathbf{0}.$$

Some known facts (some easy to show, some not):

- if  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{R}^m$  is linearly independent, then any  $\mathbf{a}_j$  cannot be a linear combination of the other  $\mathbf{a}_i$ 's; i.e.,  $\mathbf{a}_j \neq \sum_{i \neq j} \alpha_i \mathbf{a}_i$  for any  $\alpha_i$ 's.
- if  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{R}^m$  is linearly dependent, then *there exists* an  $\mathbf{a}_j$  such that  $\mathbf{a}_j$  is a linear combination of the other  $\mathbf{a}_i$ 's; i.e.,  $\mathbf{a}_j = \sum_{i \neq j} \alpha_i \mathbf{a}_i$  for some  $\alpha_i$ 's.
- if  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{R}^m$  is linearly independent, then  $n \leq m$  must hold.
- let  $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{R}^m$  be a linearly independent vector set. Suppose  $\mathbf{y} \in \operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ . Then the coefficient  $\alpha$  for the representation

$$\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{a}_i$$

is unique; i.e., there does *not* exist a  $\beta \in \mathbb{R}^n$ ,  $\beta \neq \alpha$ , such that  $\mathbf{y} = \sum_{i=1}^n \beta_i \mathbf{a}_i$ .

Let  $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{R}^m$ , and denote  $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$  as an index subset with  $k \leq n$  and  $i_j \neq i_l$  for all  $j \neq l$ .

A vector subset  $\{{\bf a}_{i_1},\ldots,{\bf a}_{i_k}\}$  is called a maximal linearly independent subset of  $\{{\bf a}_1,\ldots,{\bf a}_n\}$  if

- 1.  $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\}$  is linearly independent;
- 2.  $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\}\$  is not contained by any other linearly independent subset of  $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ .
- physical meaning: find a set of non-redundant vectors from  $\{\mathbf{a}_1, \dots \mathbf{a}_n\}$

• example:

$$\mathbf{a}_1 = \begin{bmatrix} 1\\0\\0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0\\0\\1 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 0\\1\\1 \end{bmatrix}.$$

The linearly independent subets of  $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}$  are

$$\begin{split} &\{\mathbf{a}_1\}, \{\mathbf{a}_2\}, \{\mathbf{a}_3\}, \{\mathbf{a}_4\}, \\ &\{\mathbf{a}_1, \mathbf{a}_2\}, \{\mathbf{a}_1, \mathbf{a}_3\}, \{\mathbf{a}_1, \mathbf{a}_4\}, \{\mathbf{a}_2, \mathbf{a}_3\}, \{\mathbf{a}_2, \mathbf{a}_4\}, \{\mathbf{a}_3, \mathbf{a}_4\}, \\ &\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}, \quad \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}, \quad \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}. \end{split}$$

But the maximal linearly independent subsets are

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}, \quad \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}, \quad \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}.$$

Facts:

- $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}\}$  is a maximal linearly independent subset of  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  if and only if  $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}, \mathbf{a}_j\}$  is linearly dependent for any  $j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$
- if  $\{\mathbf{a}_{i_1},\ldots,\mathbf{a}_{i_k}\}$  is a maximal linearly independent subset of  $\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$ , then

$$\operatorname{span}\{\mathbf{a}_{i_1},\ldots,\mathbf{a}_{i_k}\}=\operatorname{span}\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}.$$

#### Basis

Let  $\mathcal{S} \subseteq \mathbb{R}^m$  be a subspace with  $\mathcal{S} \neq \{\mathbf{0}\}$ .

A vector set  $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\} \subset \mathbb{R}^m$  is called a basis for S if  $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$  is linearly independent and

$$\mathcal{S} = \operatorname{span}\{\mathbf{b}_1,\ldots,\mathbf{b}_k\}.$$

• examples: let  $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\}$  be a maximal linearly independent vector subset of  $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ . Then,  $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\}$  is a basis for  $\operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ .

Some facts:

- $\bullet$  we may have more than one basis for  ${\cal S}$
- all bases for S have the same number of elements; i.e., if  $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$  and  $\{\mathbf{c}_1, \ldots, \mathbf{c}_l\}$  are bases for S, then k = l

### **Dimension of a Subspace**

The dimension of a subspace S, with  $S \neq \{0\}$ , is defined as the number of elements of a basis for S. The dimension of  $\{0\}$  is defined as 0.

- $\bullet \ \dim \mathcal{S}$  will be used as the notation for denoting the dimension of  $\mathcal{S}$
- physical meaning: effective degrees of freedom of the subspace
- examples:
  - $-\dim \mathbb{R}^m = m$
  - if k is the number of maximal linearly independent vectors of  $\{a_1, \ldots, a_n\}$ , then dim span $\{a_1, \ldots, a_n\} = k$ .

#### **Dimension of a Subspace**

**Property 1.1.** Let  $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^m$  be subspaces.

- 1. If  $S_1 \subseteq S_2$ , then  $\dim S_1 \leq \dim S_2$ .
- 2. If  $S_1 \subseteq S_2$  and  $\dim S_1 = \dim S_2$ , then  $S_1 = S_2$ .
- 3. dim S = m if and only if  $S = \mathbb{R}^m$ .
- 4.  $\dim(\mathcal{S}_1 + \mathcal{S}_2) + \dim(\mathcal{S}_1 \cap \mathcal{S}_2) = \dim \mathcal{S}_1 + \dim \mathcal{S}_2.$
- 5.  $\dim(\mathcal{S}_1 + \mathcal{S}_2) \leq \dim \mathcal{S}_1 + \dim \mathcal{S}_2$ .

# Rank

The rank of a matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , denoted by  $\operatorname{rank}(\mathbf{A})$ , is defined as the number of elements of a maximal linearly independent subset of  $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ .

- $\bullet$  or,  $\mathrm{rank}(\mathbf{A})$  is the maximum number of linearly independent columns of  $\mathbf{A}$
- $\dim \mathcal{R}(\mathbf{A}) = \mathrm{rank}(\mathbf{A})$  by definition

Facts:

- rank(A) = rank(A<sup>T</sup>), i.e., the rank of A is also the maximum number of linearly independent rows of A
- $\operatorname{rank}(\mathbf{A} + \mathbf{B}) \le \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B})$
- $rank(AB) \leq min\{rank(A), rank(B)\}$ . Also, the equality above holds if the columns of A are linearly independent and the rows of B are linearly independent.

# Rank

- $\bullet~\mathbf{A}$  is said to have
  - full column rank if the columns of A are linearly independent (more precisely, the collection of *all* columns of A is linearly independent)

\*  $\mathbf{A} \in \mathbb{R}^{m \times n}$  being of full-column rank  $\iff m \ge n, \operatorname{rank}(\mathbf{A}) = n$ 

– full row rank if the rows of  ${\bf A}$  are linearly independent

\*  $\mathbf{A} \in \mathbb{R}^{m \times n}$  being of full-row rank  $\iff m \le n, \operatorname{rank}(\mathbf{A}) = m$ 

- full rank if  $rank(\mathbf{A}) = min\{m, n\}$ ; i.e., it has either full column rank or full row rank
- rank deficient if  $rank(\mathbf{A}) < min\{m, n\}$

### **Invertible Matrices**

A square matrix A is said to be nonsingular or invertible if the columns of A are linearly independent, and singular otherwise.

• alternatively, we say A is singular if Ax = 0 for some  $x \neq 0$ .

The inverse of an invertible A, denoted by  $A^{-1}$ , is a square matrix that satisfies

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

### **Invertible Matrices**

Facts (for a nonsingular A):

- $A^{-1}$  always exists and is unique (or there are no two inverses of A)
- $\mathbf{A}^{-1}$  is nonsingular
- $AA^{-1} = I$
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ , where  $\mathbf{A}, \mathbf{B}$  are square and nonsingular
- $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$

– as a shorthand notation, we will denote  $\mathbf{A}^{-T} = (\mathbf{A}^T)^{-1}$ 

Let  $\mathbf{A} \in \mathbb{R}^{m \times m}$ . The determinant of  $\mathbf{A}$ , denoted by  $det(\mathbf{A})$ , is defined inductively.

- if m = 1,  $det(\mathbf{A}) = a_{11}$ .
- if  $m \ge 2$ , we have the following:
  - let  $\mathbf{A}_{ij} \in \mathbb{R}^{(m-1) \times (m-1)}$  be a submatrix of  $\mathbf{A}$  obtained by deleting the *i*th row and *j*th column of  $\mathbf{A}$ . Let  $c_{ij} = (-1)^{i+j} \det(\mathbf{A}_{ij})$ .
  - cofactor expansion:

$$\det(\mathbf{A}) = \sum_{j=1}^{m} a_{ij}c_{ij}, \text{ for any } i = 1, \dots, m$$
$$\det(\mathbf{A}) = \sum_{i=1}^{m} a_{ij}c_{ij}, \text{ for any } j = 1, \dots, m$$

- remark:  $c_{ij}$ 's are called the cofactors,  $det(\mathbf{A}_{ij})$ 's are called the minors

Some interpretations of determinant:

- (important) Ax = 0 for some  $x \neq 0$  if and only if det(A) = 0
- $|\det(\mathbf{A})|$  is the volume of the parallelepiped  $\mathcal{P} = \{\mathbf{y} = \sum_{i=1}^{m} \alpha_i \mathbf{a}_i \mid \alpha_i \in [0, 1] \forall i\}$



Properties:

- det(AB) = det(A) det(B) for any  $A, B \in \mathbb{R}^{m \times m}$
- $det(\mathbf{A}) = det(\mathbf{A}^T)$
- $det(\alpha \mathbf{A}) = \alpha^m det(\mathbf{A})$  for any  $\alpha \in \mathbb{R}, \mathbf{A} \in \mathbb{R}^{m \times m}$
- $det(\mathbf{A}^{-1}) = 1/det(\mathbf{A})$  for any nonsingular  $\mathbf{A}$
- $det(\mathbf{B}^{-1}\mathbf{A}\mathbf{B}) = det(\mathbf{A})$  for any nonsingular  $\mathbf{B}$
- $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \tilde{\mathbf{A}}$ , where  $\tilde{a}_{ij} = c_{ji}$  (the cofactor) for all i, j ( $\mathbf{A}$  is nonsingular)
  - remark:  $\tilde{\mathbf{A}}$  is called the adjoint of  $\mathbf{A}$

More properties:

• if  $\mathbf{A} \in \mathbb{R}^{m imes m}$  is triangular, either upper or lower,

$$\det(\mathbf{A}) = \prod_{i=1}^{m} a_{ii}$$

- proof: apply cofactor expansion inductively
- if  $\mathbf{A} \in \mathbb{R}^{m imes m}$  takes a block upper triangular form

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix},$$

where  ${f B}$  and  ${f D}$  are square (and can be of different sizes), then

$$\det(\mathbf{A}) = \det(\mathbf{B}) \det(\mathbf{D}).$$

The same result also holds when  $\mathbf{A}$  takes a block lower triangular form.

### **Vector Norms**

A function  $f : \mathbb{R}^n \to \mathbb{R}$  is called a vector norm if

- 1.  $f(\mathbf{x}) \geq 0$  for any  $\mathbf{x} \in \mathbb{R}^n$
- 2.  $f(\mathbf{x}) = 0$  if and only if  $\mathbf{x} = \mathbf{0}$
- 3.  $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$  for any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
- 4.  $f(\alpha \mathbf{x}) = |\alpha| f(\mathbf{x})$  for any  $\alpha \in \mathbb{R}$ ,  $\mathbf{x} \in \mathbb{R}^n$
- used to measure the length of a vector
- $\bullet$  we usually use the notation  $\|\cdot\|$  to denote a norm
- also used to measure the distance of two vectors, specifically, via  $\|\mathbf{x} \mathbf{y}\|$  where  $\mathbf{x}, \mathbf{y}$  are the two vectors

#### **Vector Norm**

Examples of norm:

• 2-norm or Euclidean norm: 
$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2 = (\mathbf{x}^T \mathbf{x})^{1/2}}$$

• 1-norm or Manhattan norm: 
$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$$

• 
$$\infty$$
-norm:  $\|\mathbf{x}\|_{\infty} = \max_{i=1,\dots,n} |x_i|$ 

• *p*-norm, 
$$p \ge 1$$
:  $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ 

# $\ell_p$ Function



•  $f_p$  is not a norm for 0

Let

• when  $p \to 0$ ,  $f_p$  is like the cardinality function  $\operatorname{card}(\mathbf{x}) = \sum_{i=1}^n \mathbb{1}\{x_i \neq 0\}$ , where  $\mathbb{1}\{x \neq 0\} = 1$  if  $x \neq 0$  and  $\mathbb{1}\{x \neq 0\} = 0$  if x = 0.

### **Inner Product and Angle**

The inner product of two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} y_i x_i = \mathbf{y}^T \mathbf{x}.$$

- $\mathbf{x}, \mathbf{y}$  are said to be orthogonal to each other if  $\langle \mathbf{x}, \mathbf{y} \rangle = 0$
- $\mathbf{x}, \mathbf{y}$  are said to be parallel if  $\mathbf{x} = \alpha \mathbf{y}$  for some  $\alpha$ 
  - for parallel  ${\bf x}, {\bf y}$  we have  $\langle {\bf x}, {\bf y} \rangle = \pm \| {\bf x} \|_2 \| {\bf y} \|_2$

The angle between two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  is defined as

$$\theta = \cos^{-1} \left( \frac{\mathbf{y}^T \mathbf{x}}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2} \right)$$

- $\mathbf{x}, \mathbf{y}$  are orthogonal if  $\theta = \pm \pi/2$
- $\mathbf{x}, \mathbf{y}$  are parallel if  $\theta = 0$  or  $\theta = \pm \pi$

#### **Important Inequalities for Inner Product**

Cauchy-Schwarz inequality:

 $|\mathbf{x}^T \mathbf{y}| \le \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$ 

Also, the above equality holds if and only if either  $\mathbf{x} = \alpha \mathbf{y}$  for some  $\alpha \in \mathbb{R}$  or  $\mathbf{y} = \alpha \mathbf{x}$  for some  $\alpha \in \mathbb{R}$ .

• proof: suppose  $\mathbf{y} \neq \mathbf{0}$ ; the case of  $\mathbf{y} = \mathbf{0}$  is trivial. For any  $\beta \in \mathbb{R}$ ,

$$0 \le \|\mathbf{x} - \beta \mathbf{y}\|_{2}^{2} = (\mathbf{x} - \beta \mathbf{y})^{T} (\mathbf{x} - \beta \mathbf{y}) = \|\mathbf{x}\|_{2}^{2} - 2\beta \mathbf{x}^{T} \mathbf{y} + \beta^{2} \|\mathbf{y}\|_{2}^{2} := f(\beta) \quad (*)$$

The function f is minimized when  $\beta = (\mathbf{x}^T \mathbf{y})/||\mathbf{y}||_2^2$ . Plugging this  $\beta$  back to (\*) gives  $|\mathbf{x}^T \mathbf{y}| \leq ||\mathbf{x}||_2 ||\mathbf{y}||_2$ . Equality in (\*) holds if and only if  $\mathbf{x} - \beta \mathbf{y} = \mathbf{0}$ , or equivalently,  $\mathbf{x} = \alpha \mathbf{y}$  for some  $\alpha \in \mathbb{R}$ .

#### **Important Inequalities for Inner Product**

Hölder inequality: for any  $p,q \in [1,\infty]$  such that 1/p + 1/q = 1,

 $|\mathbf{x}^T \mathbf{y}| \le \|\mathbf{x}\|_p \|\mathbf{y}\|_q,$ 

If one of the  ${\bf x}$  and  ${\bf y}$  is 0, then equality holds. If both  ${\bf x}$  and  ${\bf y}$  do not equal 0,

- for p > 1, equality holds if and only if (i)  $|x_i|^p = \alpha |y_i|^q$  for all i and for some  $\alpha \ge 0$ , and (ii)  $\operatorname{sgn}(x_i) = \beta \operatorname{sgn}(y_i)$  for all i and for some  $\beta \in \{-1, 1\}^2$
- for  $(p,q) = (1,\infty)$ , equality holds if (i)  $|y_1| = \cdots = |y_n|$ , and (ii)  $\operatorname{sgn}(x_i) = \beta \operatorname{sgn}(y_i)$  for all i and for some  $\beta \in \{-1,1\}$
- examples:

- (p,q) = (2,2): Cauchy-Schwarz inequality -  $(p,q) = (1,\infty)$ :  $|\mathbf{x}^T \mathbf{y}| \le ||\mathbf{x}||_1 ||\mathbf{y}||_\infty$ . In fact,  $|\mathbf{x}^T \mathbf{y}| \le \sum_{i=1}^n |x_i y_i| \le \max_j |y_j| (\sum_{i=1}^n |x_i|) = ||\mathbf{x}||_1 ||\mathbf{y}||_\infty$ .

<sup>2</sup>Here we define sgn(x) = 1 if x > 0, sgn(x) = 0 if x = 0, and sgn(x) = -1 if x < 0

# **Projections on Subspaces**

Let  $S \subseteq \mathbb{R}^m$  be a nonempty closed set (not necessarily a subspace). Let  $\mathbf{y} \in \mathbb{R}^m$  be given.

A projection of  $\mathbf y$  onto  $\mathcal S$  is any solution to

$$\min_{\mathbf{z}\in\mathcal{S}} \|\mathbf{z}-\mathbf{y}\|_2^2$$

- $\bullet$  a projection of y onto  ${\cal S}$  is any point that is closest to y and lies in  ${\cal S}$
- notation: if, for every  $y \in \mathbb{R}^m$ , there is always *only one* projection of y onto S, then we denote

$$\Pi_{\mathcal{S}}(\mathbf{y}) = \arg\min_{\mathbf{z}\in\mathcal{S}} \|\mathbf{z} - \mathbf{y}\|_2^2$$

and  $\Pi_{\mathcal{S}}$  is called *the* projection (or projection operator) of y onto  $\mathcal{S}$ .

#### **Projections onto Subspaces**

**Theorem 1.2** (Projection Theorem). Let S be a subspace of  $\mathbb{R}^m$ .

- 1. for every  $\mathbf{y} \in \mathbb{R}^m$ , there exists a unique vector  $\mathbf{y}_s \in \mathcal{S}$  that minimizes  $\|\mathbf{z} \mathbf{y}\|_2^2$ over  $\mathbf{z} \in \mathcal{S}$ . Thus, we can use the notation  $\Pi_{\mathcal{S}}(\mathbf{y}) = \arg\min_{\mathbf{z} \in \mathcal{S}} \|\mathbf{z} - \mathbf{y}\|_2^2$ .
- 2. given  $\mathbf{y} \in \mathbb{R}^m$ , we have the equivalence

$$\mathbf{y}_s = \Pi_{\mathcal{S}}(\mathbf{y}) \quad \iff \quad \mathbf{y}_s \in \mathcal{S}, \quad \mathbf{z}^T(\mathbf{y}_s - \mathbf{y}) = 0 \text{ for all } \mathbf{z} \in \mathcal{S}.$$

- a special case of the projection theorem for convex sets
  - the latter plays a key role in convex optimization
- the subspace projection theorem above is very useful, as we will see

**Projections onto Subspaces** 



### **Orthogonal Complements**

Let  $\mathcal{S} \subseteq \mathbb{R}^m$  be a nonempty closed set.

The orthogonal complement of  ${\mathcal S}$  is defined as

$$\mathcal{S}^{\perp} = \{ \mathbf{y} \in \mathbb{R}^m \mid \mathbf{z}^T \mathbf{y} = 0 \text{ for all } \mathbf{z} \in \mathcal{S} \}.$$

- $\mathcal{S}^{\perp}$  is a subspace (easy to verify)
- any  $\mathbf{z} \in \mathcal{S}, \mathbf{y} \in \mathcal{S}^{\perp}$  are orthogonal
- either  $\mathcal{S} \cap \mathcal{S}^{\perp} = \{\mathbf{0}\}$  or  $\mathcal{S} \cap \mathcal{S}^{\perp} = \emptyset$
- some facts for subspaces:

- 
$$\mathcal{R}(\mathbf{A})^{\perp} = \mathcal{N}(\mathbf{A}^T)$$
 (also easy to verify)  
-  $\mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^T)^{\perp}$ 

# **Orthogonal Complements**

What happens to the orthogonal complement if  $\mathcal{S}$  is a subspace?

**Theorem 1.3.** Let  $\mathcal{S} \subseteq \mathbb{R}^m$  be a subspace.

1. for every  $\mathbf{y} \in \mathbb{R}^m$ , there exists a unique  $(\mathbf{y}_s, \mathbf{y}_c) \in \mathcal{S} imes \mathcal{S}^\perp$  such that

$$\mathbf{y} = \mathbf{y}_s + \mathbf{y}_c.$$

Also, such a  $(\mathbf{y}_s, \mathbf{y}_c)$  is  $\mathbf{y}_s = \Pi_{\mathcal{S}}(\mathbf{y}), \mathbf{y}_c = \mathbf{y} - \Pi_{\mathcal{S}}(\mathbf{y}).$ 

- 2. the projection of y onto  $S^{\perp}$  can be determined by  $\Pi_{S^{\perp}}(\mathbf{y}) = \mathbf{y} \Pi_{S}(\mathbf{y})$ .
- proof sketch: by the projection theorem. We can rephrase the projection theorem as

$$\mathbf{y}_s \in \mathcal{S}, \ \mathbf{y} - \mathbf{y}_s \in \mathcal{S}^{\perp} \quad \Longleftrightarrow \quad \mathbf{y}_s \in \Pi_{\mathcal{S}}(\mathbf{y}).$$

This leads us to Statement 1 of Theorem 1.3.

# **Orthogonal Complements**

Consequences of Theorem 1.3:

**Property 1.2.** Let  $\mathcal{S} \subseteq \mathbb{R}^m$  be a subspace.

- 1.  $S + S^{\perp} = \mathbb{R}^m$ ;
- 2. dim  $\mathcal{S}$  + dim  $\mathcal{S}^{\perp}$  = m;
- 3.  $(\mathcal{S}^{\perp})^{\perp} = \mathcal{S}.$
- examples: let  $\mathbf{A} \in \mathbb{R}^{m \times n}$ .
  - $-\dim \mathcal{R}(\mathbf{A}) + \dim \mathcal{R}(\mathbf{A})^{\perp} = m$
  - and then  $\dim \mathcal{R}(\mathbf{A}) + \dim \mathcal{N}(\mathbf{A}^T) = m$
  - and then  $\dim \mathcal{N}(\mathbf{A}) = n \dim \mathcal{R}(\mathbf{A}^T) = n \operatorname{rank}(\mathbf{A}) \ge n \min\{m, n\}$ 
    - \* implication: if  ${\bf A}$  is fat, the dim. of  ${\cal N}({\bf A})$  is at least n-m

**Property 1.3.**  $S_1 \cap S_2 = (S_1^{\perp} + S_2^{\perp})^{\perp}$ , where  $S_1, S_2 \subseteq \mathbb{R}^m$  are subspaces.

A collection of nonzero vectors  $\mathbf{a}_1,\ldots,\mathbf{a}_n\in\mathbb{R}^m$  is said to be

- orthogonal if  $\mathbf{a}_i^T \mathbf{a}_j = 0$  for all i, j with  $i \neq j$
- orthonormal if  $\|\mathbf{a}_i\|_2 = 1$  for all i and  $\mathbf{a}_i^T \mathbf{a}_j = 0$  for all i, j with  $i \neq j$ .

The same definition applies to complex  $\mathbf{a}_i$ 's, but we need to replace "T" with "H". Examples:

- $\{\mathbf{e}_1,\ldots,\mathbf{e}_m\}\subset\mathbb{R}^m$  is orthonormal; in fact, it's an orthonormal basis for  $\mathbb{R}^m$
- any subset of  $\{\mathbf{e}_1,\ldots,\mathbf{e}_m\}$  is orthornormal
- (to be learnt) discrete Fourier transform (DFT), Haar transform, etc., form orthonormal bases

Some immediate facts:

- an orthonormal set of vectors is also linearly independent.
- let  $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{R}^m$  be an orthonormal set of vectors. Suppose  $\mathbf{y} \in \operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ . Then the coefficient  $\alpha$  for the representation

$$\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{a}_i$$

is uniquely given by  $\alpha_i = \mathbf{a}_i^T \mathbf{y}$ ,  $i = 1, \dots, n$ .

A not so immediate fact:

- (important) every subspace S with  $S \neq \{0\}$  has an orthonormal basis.
  - this will be clear when we consider Gram-Schmidt

A real matrix  ${\bf Q}$  is said to be

- orthogonal if it is square and its columns are orthonormal (why we call it an orthogonal matrix, but not an orthonormal matrix?)
- semi-orthogonal if its columns are orthonormal
  - a semi-orthogonal  ${\bf Q}$  must be tall or square

A complex matrix  $\mathbf{Q}$  is said to be unitary if it is square and its columns are orthonormal, and semi-unitary if its columns are orthonormal.

Example: consider a transformation  $\mathbf{y}=\mathbf{Q}\mathbf{x}$  , where

$$\mathbf{Q} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

where  $\theta \in [0, 2\pi)$ . This **Q** is orthogonal. Also, it performs rotation and reflection.

Facts:

- $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$  and  $\mathbf{Q} \mathbf{Q}^T = \mathbf{I}$  for orthogonal  $\mathbf{Q}$
- $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$  (but not necessarily  $\mathbf{Q} \mathbf{Q}^T = \mathbf{I}$ ) for semi-orthogonal  $\mathbf{Q}$
- $\|\mathbf{Q}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$  for orthogonal  $\mathbf{Q}$ 
  - physical meaning: rotation and reflection do not affect the vector length
- for every tall and semi-orthogonal matrix  $\mathbf{Q}_1 \in \mathbb{R}^{n \times k}$ , there exists a matrix  $\mathbf{Q}_2 \in \mathbb{R}^{n \times (n-k)}$  such that  $[\mathbf{Q}_1 \mathbf{Q}_2]$  is orthogonal

Question: given a subspace S, how do we know that it has an orthonormal basis?

- we know that every subspace has a basis, c.f. Theorem 1.1
- but the theorem doesn't say if that basis is orthonormal
- we can construct an orthonormal basis from a basis—and one way to do it is the Gram-Schmidt procedure

Algorithm: Gram-Schmidt  
input: a collection of vectors 
$$\mathbf{a}_1, \ldots, \mathbf{a}_n$$
, presumably linearly independent  
 $\tilde{\mathbf{q}}_1 = \mathbf{a}_1, \, \mathbf{q}_1 = \tilde{\mathbf{q}}_1 / \|\tilde{\mathbf{q}}_1\|_2$   
for  $i = 2, \ldots, n$   
 $\tilde{\mathbf{q}}_i = \mathbf{a}_i - \sum_{j=1}^{i-1} (\mathbf{q}_j^T \mathbf{a}_i) \mathbf{q}_j$   
 $\mathbf{q}_i = \tilde{\mathbf{q}}_i / \|\tilde{\mathbf{q}}_i\|_2$   
end  
output:  $\mathbf{q}_1, \ldots, \mathbf{q}_n$ 

• Fact: Suppose that  $\mathbf{a}_1, \ldots, \mathbf{a}_n$  are linearly independent. The collection of vectors  $\mathbf{q}_1, \ldots, \mathbf{q}_n$  produced by the Gram-Schmidt procedure is orthonormal and satisfies

$$\operatorname{span}\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}=\operatorname{span}\{\mathbf{q}_1,\ldots,\mathbf{q}_n\}.$$

 here we use Gram-Schmidt to identify the existence of an orthonormal basis for a subspace, but it is a numerical algorithm

Proof of the fact on the last page:

- assume linearly independent  $\mathbf{a}_1,\ldots,\mathbf{a}_n$
- consider i = 2.

–  $\tilde{\mathbf{q}}_2$  is a linear combination of  $\mathbf{a}_1, \mathbf{a}_2$  and is nonzero:

$$\tilde{\mathbf{q}}_2 = \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2) \mathbf{q}_1 = \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2 / \|\mathbf{a}_1\|_2) \mathbf{a}_1;$$
 (†)

the linear independence of  $\mathbf{a}_1, \mathbf{a}_2$  implies  $\tilde{\mathbf{q}}_2 \neq \mathbf{0}$ .

- $\mathbf{a}_2$  is a linear combination of  $\mathbf{q}_1, \mathbf{q}_2$ : seen from (†)
- consequence:  $\operatorname{span}\{\mathbf{a}_1,\mathbf{a}_2\} = \operatorname{span}\{\mathbf{q}_1,\mathbf{q}_2\}$  (why?)
- $\tilde{\mathbf{q}}_2$  is orthogonal to  $\mathbf{q}_1$ :

$$\mathbf{q}_1^T \tilde{\mathbf{q}}_2 = \mathbf{q}_1^T (\mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2) \mathbf{q}_1) = \mathbf{q}_1^T \mathbf{a}_2 - \mathbf{q}_1^T \mathbf{a}_2 = 0.$$

- consider  $i \geq 2$ .
  - $\tilde{\mathbf{q}}_i$  is a linear combination of  $\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}$  and is nonzero: by induction,  $\mathbf{q}_1, \ldots, \mathbf{q}_{i-1}$  are linear combinations of  $\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}$ . So,

$$\tilde{\mathbf{q}}_i = \mathbf{a}_i - \sum_{j=1}^{i-1} (\mathbf{q}_j^T \mathbf{a}_i) \mathbf{q}_j$$
(‡)

is a linear combination of  $\mathbf{a}_1, \ldots, \mathbf{a}_i$ . The linear independence of  $\mathbf{a}_1, \ldots, \mathbf{a}_i$  implies  $\tilde{\mathbf{q}}_i \neq \mathbf{0}$ .

- $\mathbf{a}_i$  is a linear combination of  $\mathbf{q}_1, \ldots, \mathbf{q}_i$ : seen from (‡)
- consequence:  $\operatorname{span}\{\mathbf{a}_1,\ldots,\mathbf{a}_i\}=\operatorname{span}\{\mathbf{q}_1,\ldots,\mathbf{q}_i\}$  (why?)
- $\tilde{\mathbf{q}}_i$  is orthogonal to  $\mathbf{q}_1, \ldots, \mathbf{q}_{i-1}$ : by induction,  $\mathbf{q}_1, \ldots, \mathbf{q}_{i-1}$  are orthonormal. For any  $k \in \{1, \ldots, i-1\}$ ,

$$\mathbf{q}_k^T \tilde{\mathbf{q}}_i = \mathbf{q}_k^T (\mathbf{a}_i - \sum_{j=1}^{i-1} (\mathbf{q}_j^T \mathbf{a}_i) \mathbf{q}_j) = \mathbf{q}_k^T \mathbf{a}_i - \mathbf{q}_k^T \mathbf{a}_i = 0.$$

More comments:

• the step

$$\tilde{\mathbf{q}}_i = \mathbf{a}_i - \sum_{j=1}^{i-1} (\mathbf{q}_j^T \mathbf{a}_i) \mathbf{q}_j$$

can be shown to be equivalent to

$$\tilde{\mathbf{q}}_i = \Pi_{\operatorname{span}\{\mathbf{q}_1,\dots,\mathbf{q}_{i-1}\}^{\perp}}(\mathbf{a}_i) = \Pi_{\operatorname{span}\{\mathbf{a}_1,\dots,\mathbf{a}_{i-1}\}^{\perp}}(\mathbf{a}_i);$$

this will be seen in the LS lecture.

- the Gram-Schmidt procedure can be modified in various ways
  - e.g., it can be modified to do linear independence test, or to find a maximal linearly independent vector subset

### **Matrix Product Representations**

Let  $\mathbf{A} \in \mathbb{R}^{m imes k}$ ,  $\mathbf{B} \in \mathbb{R}^{k imes n}$ , and consider

 $\mathbf{C} = \mathbf{AB}.$ 

• column representation:

$$\mathbf{c}_i = \mathbf{A}\mathbf{b}_i, \quad i = 1, \dots, n$$

(I didn't say anything so I assume you know that  $c_i$  and  $b_i$  are the *i*th column of C and B, resp.)

• inner-product representation: redefine  $\mathbf{a}_i \in \mathbb{R}^k$  as the *i*th row of  $\mathbf{A}$ .

(

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{b}_1 & \cdots & \mathbf{a}_1^T \mathbf{b}_n \\ \vdots & & \vdots \\ \mathbf{a}_m^T \mathbf{b}_1 & \cdots & \mathbf{a}_m^T \mathbf{b}_n \end{bmatrix}$$

Thus,

$$c_{ij} = \mathbf{a}_i^T \mathbf{b}_j, \quad \text{for any } i, j.$$

### **Matrix Product Representations**

• outer-product representation: redefine  $\mathbf{b}_i \in \mathbb{R}^k$  as the *i*th row of **B**.

$$\mathbf{C} = \mathbf{A}(\mathbf{I})\mathbf{B} = \mathbf{A}\left(\sum_{i=1}^{k} \mathbf{e}_{i}\mathbf{e}_{i}^{T}\right)\mathbf{B} = \sum_{i=1}^{k} \mathbf{A}\mathbf{e}_{i}\mathbf{e}_{i}^{T}\mathbf{B}$$

Thus,

$$\mathbf{C} = \sum_{i=1}^k \mathbf{a}_i \mathbf{b}_i^T$$



### **Matrix Product Representations**

- a matrix of the form  $\mathbf{X} = \mathbf{a}\mathbf{b}^T$  for some  $\mathbf{a}, \mathbf{b}$  is called a rank-one outer product. It can be verified that  $\operatorname{rank}(\mathbf{X}) \leq 1$ , and  $\operatorname{rank}(\mathbf{X}) = 1$  if  $\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$ .
- the outer-product representation  $\mathbf{C} = \sum_{i=1}^{k} \mathbf{a}_i \mathbf{b}_i^T$  is a sum of k rank-one outer products
- does it mean that  $rank(\mathbf{C}) = k$ ?

- 
$$\operatorname{rank}(\mathbf{C}) \leq \sum_{i=1}^{k} \operatorname{rank}(\mathbf{a}_i \mathbf{b}_i^T) \leq k$$
 is true <sup>3</sup>

- but the above equality is generally not attained; e.g., k=2,  ${\bf a}_1={\bf a}_2$ ,  ${\bf b}_1=-{\bf b}_2$  leads to  ${\bf C}={\bf 0}$
- $rank(\mathbf{C}) = k$  only when **A** has full-column rank and **B** has full-row rank (requires a proof)

<sup>3</sup>use the rank inequality  $rank(A + B) \leq rank(A) + rank(B)$ .

#### **Block Matrix Manipulations**

Sometimes it may be useful to manipulate matrices in a block form.

• let  $\mathbf{A} \in \mathbb{R}^{m imes n}$ ,  $\mathbf{x} \in \mathbb{R}^n$ . By partitioning

$$\mathbf{A} = egin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{x} = egin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

where  $\mathbf{A}_1 \in \mathbb{R}^{m \times n_1}, \mathbf{A}_2 \in \mathbb{R}^{m \times n_2}, \mathbf{x}_1 \in \mathbb{R}^{n_1}, \mathbf{x}_2 \in \mathbb{R}^{n_2}$ , with  $n_1 + n_2 = n$ , we can write

$$\mathbf{A}\mathbf{x} = \mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2\mathbf{x}_2$$

• similarly, by partitioning

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix},$$

we can write

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 \\ \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 \end{bmatrix}$$

### **Block Matrix Manipulations**

• consider AB. By an appropriate partitioning,

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \mathbf{A}_1 \mathbf{B}_1 + \mathbf{A}_2 \mathbf{B}_2$$

• similarly, by an appropriate partitioning,

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \mathbf{B}_1 & \mathbf{A}_1 \mathbf{B}_2 \\ \mathbf{A}_2 \mathbf{B}_1 & \mathbf{A}_2 \mathbf{B}_2 \end{bmatrix}$$

 we showcase two-block partitioning only, but the same manipulations apply to multi-block partitioning like

$$\mathbf{A} = egin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1q} \ dots & & dots \ \mathbf{A}_{p1} & \cdots & \mathbf{A}_{pq} \end{bmatrix}$$

#### **Extension to** $\mathbb{C}^n$

- all the concepts described above apply to the complex case
- we only need to replace every " $\mathbb{R}$ " with " $\mathbb{C}$ ", and every "T" with "H"; e.g.,

span{
$$\mathbf{a}_1,\ldots,\mathbf{a}_n$$
} = { $\mathbf{y} \in \mathbb{C}^m \mid \mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{a}_i, \ \boldsymbol{\alpha} \in \mathbb{C}^n$ },

 $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^H \mathbf{x}$ ,  $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^H \mathbf{x}}$ , and so forth.

#### Extension to $\mathbb{R}^{m \times n}$

- the concepts also apply to the matrix case
  - e.g., we may write

span{
$$\mathbf{A}_1, \dots, \mathbf{A}_k$$
} = { $\mathbf{Y} \in \mathbb{R}^{m \times n} \mid \mathbf{Y} = \sum_{i=1}^k \alpha_i \mathbf{A}_i, \ \boldsymbol{\alpha} \in \mathbb{R}^k$ }.

- sometimes it is more convenient to *vectorize* X as a vector  $x \in \mathbb{R}^{mn}$ , and use the same treatment as in the  $\mathbb{R}^n$  case
- inner product for  $\mathbb{R}^{m \times n}$ :

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} y_{ij} = \operatorname{tr}(\mathbf{Y}^T \mathbf{X}),$$

- the matrix version of the Euclidean norm is called the Frobenius norm:

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |x_{ij}|^2} = \sqrt{\operatorname{tr}(\mathbf{X}^T \mathbf{X})}$$

• extension to  $\mathbb{C}^{m imes n}$  is just as straightforward as in that to  $\mathbb{C}^n$ 

- every vector/matrix operation such as x + y, y<sup>T</sup>x, Ax, ... incurs computational costs, and they cost more as the vector and matrix sizes get bigger
- we typically look at floating point arithmetic operations, such as add, subtract, multiply, and divide

- flops: one flop means one floating point arithmetic operation.
- flop counts of some standard vector/matrix operations:

for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times p}$ ,

- $\mathbf{x} + \mathbf{y}$ : n adds, so n flops
- $\mathbf{y}^T \mathbf{x}$ : *n* multiplies and n-1 adds, so 2n-1 flops
- Ax: m inner products, so m(2n-1) flops
- AB: do "Ax" above p times, so pm(2n-1) flops

- we are often interested in the *order* of the complexity
- big O notation: given two functions f(n), g(n), the notation

$$f(n) = \mathcal{O}(g(n))$$

means that there exists a constant C > 0 and  $n_0$  such that  $|f(n)| \le C|g(n)|$  for all  $n \ge n_0$ .

- big O complexities of standard vector/matrix operations:
  - $\mathbf{x} + \mathbf{y}$ :  $\mathcal{O}(n)$  flops
  - $\mathbf{y}^T \mathbf{x}$ :  $\mathcal{O}(n)$  flops
  - Ax:  $\mathcal{O}(mn)$  flops
  - AB:  $\mathcal{O}(mnp)$  flops
  - (we'll learn it later) solve  $\mathbf{y} = \mathbf{A}\mathbf{x}$  for  $\mathbf{x}$ , with  $\mathbf{A} \in \mathbb{R}^{n \times n}$ :  $\mathcal{O}(n^3)$  flops

- big O complexities are commonly used, although we should be careful sometimes
- example: suppose you have an algorithm whose exact flop count is

$$f(n) = 3n^3 + 8n^2 + 2n + 1234.$$

–  $\mathcal{O}(n^3)$  flops

- big O makes sense for large n;  $n^3$  dominates as n is large
- but be careful: for small  $n,\,{\rm it's}\;1234$  that consumes more
- example: suppose you have two algorithms for the same problem. Their exact flop counts are

$$f_1(n) = n^3$$
,  $f_2(n) = \frac{1}{2}n^3$ .

- their big O complexities are the same:  $\mathcal{O}(n^3)$
- but two times faster is two times faster!

- example: suppose our algorithm deals with complex vector and matrix operations. Define one flop as one real flop.
  - one complex add = 2 real adds = 2 flops
  - one complex multiply = 4 real multiplies + 2 real adds = 6 flops
     :

When we report big O complexity, the scaling factors above are not seen

#### **Exercise: Count the Complexity of Gram Schmidt**

• recall the Gram-Schmidt procedure recursively computes

$$\tilde{\mathbf{q}}_i = \mathbf{a}_i - \sum_{j=1}^{i-1} (\mathbf{q}_j^T \mathbf{a}_i) \mathbf{q}_j, \quad \mathbf{q}_i = \tilde{\mathbf{q}}_i / \|\tilde{\mathbf{q}}_i\|_2, \quad i = 1, \dots, n.$$

• consider iteration *i*.

- every 
$$\mathbf{q}_j^T \mathbf{a}_i$$
,  $j = 1, \dots, i = 1$ , takes  $\mathcal{O}(m)$ 

- then, computing  $\tilde{\mathbf{q}}_i = \mathbf{a}_i \sum_{j=1}^{i-1} (\mathbf{q}_j^T \mathbf{a}_i) \mathbf{q}_j$  is almost the same as the operation "Ax"; it takes  $\mathcal{O}(mi)$
- $\tilde{\mathbf{q}}_i = \tilde{\mathbf{q}}_i / \|\tilde{\mathbf{q}}_i\|_2$  requires  $\mathcal{O}(m)$  (one divide, one  $\sqrt{\cdot}$ , one inner product  $\tilde{\mathbf{q}}_i^T \tilde{\mathbf{q}}_i$ )
- total complexity for iteration i:  $(i-1) \times \mathcal{O}(m) + \mathcal{O}(mi) + \mathcal{O}(mi) = \mathcal{O}(mi)$
- total complexity of the whole algorithm:

$$\mathcal{O}(m\sum_{i=1}^{n}i) = \mathcal{O}(m\frac{n(n+1)}{2}) = \mathcal{O}(mn^2)$$

- Discussion: flop counts do not always translate into the actual efficiency of the execution of an algorithm, say, in terms of actual running time.
- things like pipelining, FPGA, parallel computing (multiple GPUs, multiple servers, cloud computing), etc., can make the story different.
- flop counts also ignore memory usage and other overheads...
- that said, we need at least a crude measure of how computationally costly an algorithm would be, and counting the flops serves that purpose.

- computational complexities depend much on how we design and write an algorithm
- generally, it is about
  - top-down, analysis-guided, designs: often seen in class, often look elegant
  - street-smart, possibly bottom-up, tricks: usually not taught much in class, also not commonplace in papers (unless you download and read somebody's code), subtly depends on your problem at hand, but a bunch of small differences can make a big difference, say in actual running time
- here we give several, but by no means all, tips for saving computations

- apply matrix operations wisely
- example: try this on MATLAB

```
>> A=randn(5000,2); B=randn(2,10000); C=randn(10000,10000);
>>
>> tic; D= A*B*C; toc
Elapsed time is 12.238567 seconds.
>> tic; D= (A*B)*C; toc % ask MATLAB to do AB first
Elapsed time is 12.640961 seconds.
>> tic; D= A*(B*C); toc % ask MATLAB to do BC first
Elapsed time is 0.222270 seconds.
```

- let us analyze the complexities in the last example
  - $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times p}$ ,  $\mathbf{C} \in \mathbb{R}^{p \times p}$ , with  $n \ll \min\{m, p\}$ . We want to compute  $\mathbf{D} = \mathbf{ABC}$ .
  - if we compute AB first, and then D = (AB)C, the flop count will be

$$\mathcal{O}(mnp) + \mathcal{O}(mp^2) = \mathcal{O}(m(n+p)p) \approx \mathcal{O}(mp^2)$$

– if we compute  ${\bf BC}$  first, and then  ${\bf D}={\bf A}({\bf BC}),$  the flop count will be

$$\mathcal{O}(np^2) + \mathcal{O}(mnp) = \mathcal{O}((m+p)np).$$

– the 2nd option is preferable if n is much smaller than m, p

- use structures, if available
- example: let  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and suppose that

$$a_{ij} = 0$$
 for all  $i, j$  such that  $|i - j| > p$ ,

- for some integer p > 0.
- such a structured  ${\bf A}$  is called band diagonal
- if we don't use structures, computing  $\mathbf{A}\mathbf{x}$  requires  $\mathcal{O}(n^2)$



- if we use the band diagonal structures, we can compute  $\mathbf{A}\mathbf{x}$  with  $\mathcal{O}(pn)$ 

- use sparsity, if available
- a vector or matrix is said to be sparse if it contains many zero elements
  - we assume unstructured sparsity



- $\bullet$  let  ${\rm nnz}({\bf x})$  denote the number of nonzero elements of a vector  ${\bf x};$  the same notation applies to matrices
- flop counts: for  $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ ,  $\mathbf{A}\in\mathbb{R}^{m imes n}$ ,  $\mathbf{B}\in\mathbb{R}^{n imes p}$ ,
  - $\mathbf{x} + \mathbf{y}: \text{ from } 0 \text{ and } \min\{ \operatorname{nnz}(\mathbf{x}), \operatorname{nnz}(\mathbf{y}) \} \text{ flops} \Longrightarrow \mathcal{O}(\min\{ \operatorname{nnz}(\mathbf{x}), \operatorname{nnz}(\mathbf{y}) \})$
  - $\mathbf{y}^T \mathbf{x}$ : from 0 to  $2\min\{\max(\mathbf{x}), \max(\mathbf{y})\}$  flops  $\Longrightarrow \mathcal{O}(\min\{\max(\mathbf{x}), \max(\mathbf{y})\})$
  - Ax, x being dense: from  $\mathrm{nnz}(\mathbf{A})$  to  $2\mathrm{nnz}(\mathbf{A})$  flops  $\Longrightarrow \mathcal{O}(\mathrm{nnz}(\mathbf{A}))$
  - AB: no simple expression for the flops, but at most  $2\min\{\max(\mathbf{A})p, \max(\mathbf{B})m\}$  flops  $\Longrightarrow \mathcal{O}(\min\{\max(\mathbf{A})p, \max(\mathbf{B})m\})$
- reference: S. Boyd and L. Vandenberghe, Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares, 2018. Available online at https://web.stanford.edu/~boyd/ vmls/vmls.pdf.