ELEG 2310B: Principles of Communication Systems	2021-22 First Term

Handout 9: Pulse Modulation

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Suggested Reading: Chapter 7 of Simon Haykin and Michael Moher, *Communication Systems (5th Edition)*, Wily & Sons Ltd; or Chapter 6 of B. P. Lathi and Z. Ding, *Modern Digital and Analog Communication Systems (4th Edition)*, Oxford University Press.

In previous handouts, we have been focusing on modulation schemes where a sinusoidal (and continuous) carrier wave is used to carry information. This handout considers a different class of modulation schemes where pulses are used to carry information.

1 Pulse Modulation Schemes

Our study begins with the *pulse-amplitude modulation (PAM)* scheme. More precisely, we consider analog PAM where we wish to modulate an arbitrary message signal $m(t)$.

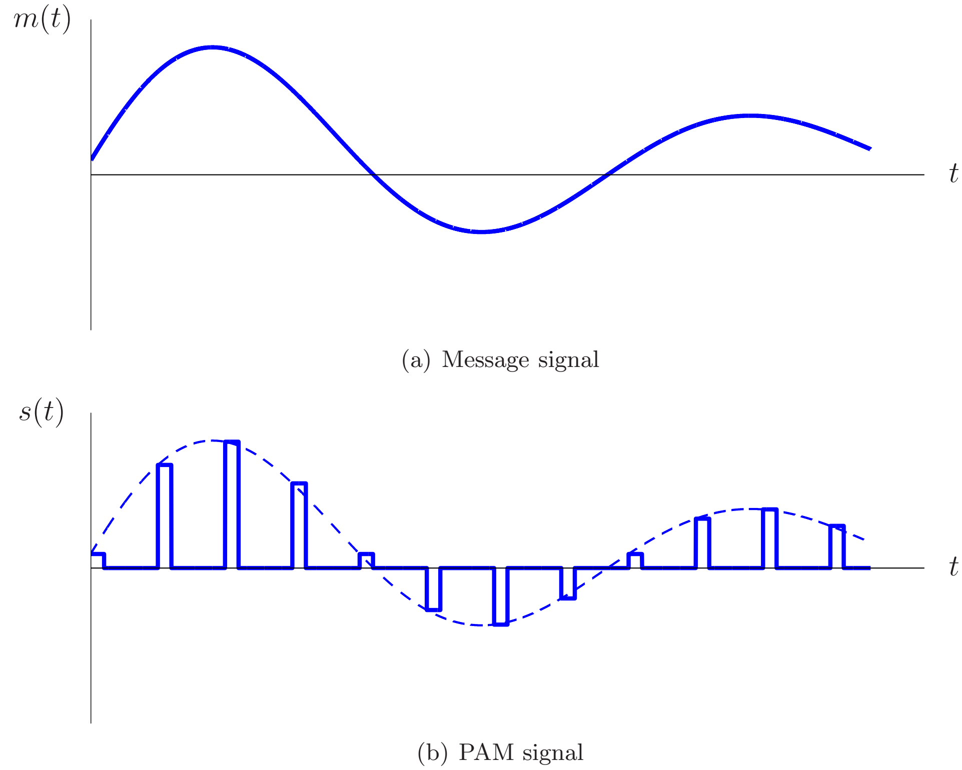


Figure 1: A waveform illustration of PAM.

Figure 1 illustrates how a PAM waveform looks like. There are two operations for generating the PAM signal. First, a *sampling process* is applied to the message signal $m(t)$. The sampling process samples $m(t)$ every T_s seconds in an instantaneous and time-uniform manner, resulting in a sequence of discrete-time samples $\{m(nT_s)\}_{n=-\infty}^{\infty}$. Second, each sample $m(nT_s)$ is carried by a pulse. The amplitude of the pulse is used to carry the value of $m(nT_s)$, and the time instant for the pulse transmission is $t = nT_s$ (i.e., the same time instant as the sampling time instant of $m(nT_s)$). Typically, each pulse is a rectangular pulse whose duration is shorter than the period T_s. The two operations described above can be accomplished in real world by digital circuit technology, specifically, "sample-and-hold" circuits.

In terms of applications, the digital version of PAM (to be further studied later) appear quite frequently in old and some existing baseband wireline communication systems, e.g., telephone lines, and local area network (LAN) lines for computer communication.

The signal representation for PAM is as follows. Based on the above description, we can express the PAM signal as

$$s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s), \tag{1}$$

where T_s is called the *sampling period*, and $h(t)$ denotes the pulse. The pulse $h(t)$ takes the form

$$h(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where T is the pulse duration. Also, it is assumed that $T < T_s$. As a further remark, it should be noted that while we will consider the rectangular pulse in this handout, other pulse shapes may also be used *in principle*.

In addition to PAM, there are other pulse modulation schemes. One is *pulse-duration modulation (PDM)*, which is also called *pulse-width modulation* or *pulse-length modulation*. PDM employs the same sampling process as in PAM, but uses pulse durations to represent (and send) the values of the samples $\{m(nT_s)\}_{n=-\infty}^{\infty}$. An illustration is given in Figure 2 to show how the message signal and the corresponding PDM waveform are related. PDM is not commonly seen in modern communication systems, but it is used quite extensively in a non-communication context, namely, DC motor control.

Another pulse modulation scheme is *pulse-position modulation (PPM)*. Again, PPM employs the same sampling process as in PAM. What is different in PPM is that PPM uses pulse positions to carry information; this is illustrated in Figure 2.(c). Specifically, the PPM signal may be represented by the following formula

$$s(t) = \sum_{n=-\infty}^{\infty} h(t - nT_s - k_p m(nT_s)), \tag{3}$$

where T_s is again the sampling period, and k_p is the sensitivity of the pulse-position modulator. The sensitivity constant k_p should satisfy

$$k_p |m(t)|_{\max} < \frac{T_s}{2}$$

such that the transmitted pulses do not overlap each other in time. PPM (more precisely, its digital version) has been considered in a kind of communication systems called ultra-wideband communication.

We will focus on PAM. In particular, we ask ourselves the following questions: What is the spectrum and bandwidth requirement of PAM? Can the original message signal be exactly recovered, or demodulated, at the receiver side? What are the underlying conditions for exact recovery?

Note that similar questions have also been asked, and addressed, in our previous study of carrier-modulation schemes.

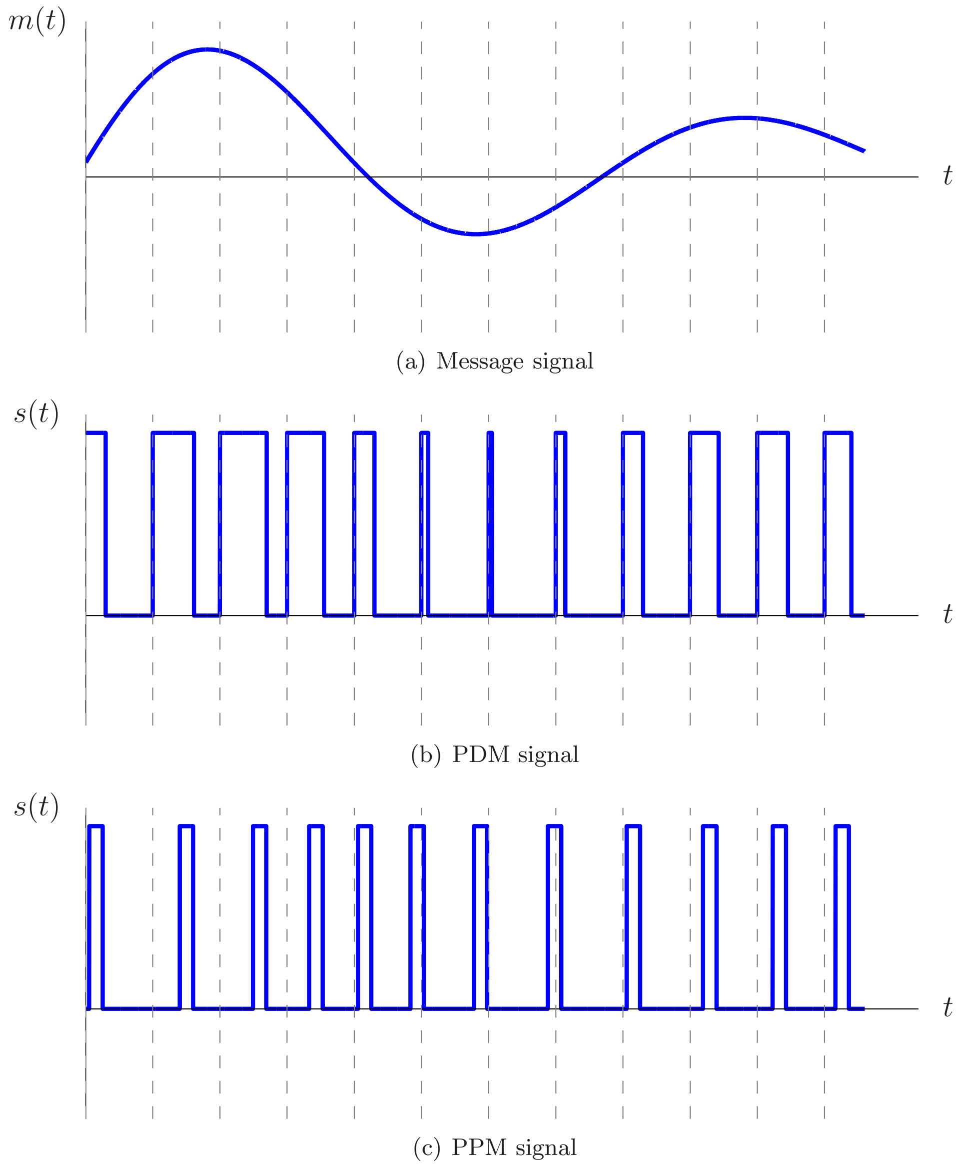


Figure 2: A waveform illustration of PDM and PPM. The grey vertical lines mark the sampling time instants.

2 Sampling Theorem

This section reviews the sampling theorem. As will be seen later, the proof of the sampling theorem has a strong link to the questions of PAM spectrum and exact recovery.

The problem setup is as follows. Suppose that we have an arbitrary signal $g(t)$. We apply time-uniform sampling to $g(t)$, with a sampling period T_s, to obtain a sequence of samples $\{g(nT_s)\}_{n=-\infty}^{\infty}$. Figures 3.(a) and 3.(c) show an illustration of the sampling process. The question arising is whether the signal $g(t)$ can be perfectly reconstructed from its samples $\{g(nT_s)\}_{n=-\infty}^{\infty}$.

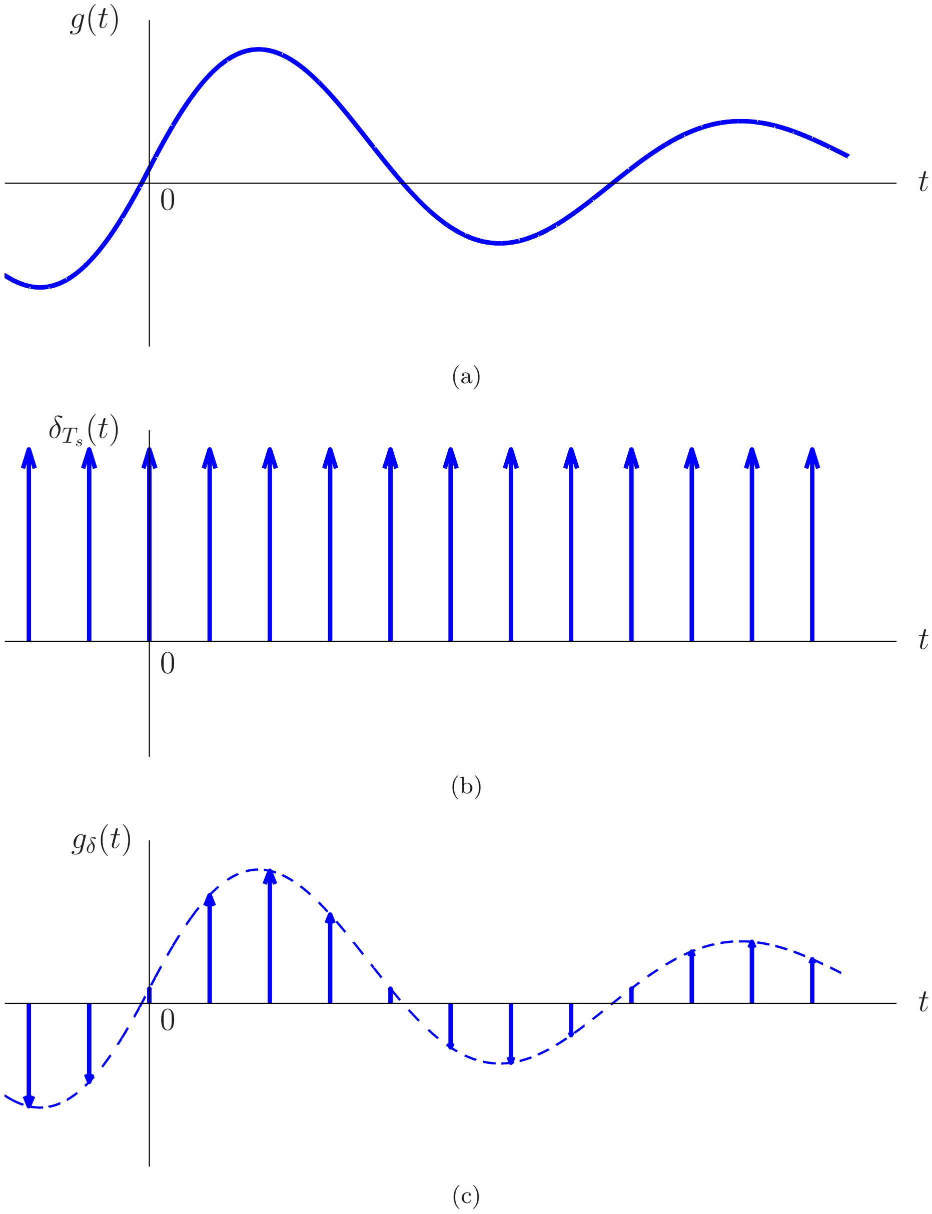


Figure 3: A waveform illustration of the sampling process.

To study the question, we consider an *ideal sampled signal*

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \tag{4}$$

which is a continuous-time representation of the samples. The signal $g_\delta(t)$ is illustrated in Figure 3.(c). Our first step is to derive the Fourier transform of $g_\delta(t)$. The ideal sampled signal can be rewritten as

$$g_\delta(t) = g(t) \cdot \delta_{T_s}(t), \tag{5}$$

where

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{6}$$

is a periodic train of Dirac delta impulses (note that in the reformulation from (4) to (5), we have applied a basic property of the Dirac delta impulse function, namely, that $x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)$ for a continuous $x(t)$). Since $\delta_{T_s}(t)$ is a periodic signal with period T_s, we can apply the Fourier series representation to obtain

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T_s}}, \tag{7}$$

where

$$c_k = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta_{T_s}(t) e^{-j\frac{2\pi kt}{T_s}}\, dt = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-j\frac{2\pi kt}{T_s}}\, dt \tag{8}$$

are the respective Fourier series coefficients. It can be easily shown that the integral on the right-hand side of (8) always equals 1 (again, we use the property $x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)$). As a result, the Fourier series coefficients are simply

$$c_n = \frac{1}{T_s}, \quad \text{for all } n. \tag{9}$$

By substituting the above result into (7), and then putting the resulting (7) into (5), we have the following expression for $g(t)$

$$g_\delta(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} g(t) e^{j\frac{2\pi kt}{T_s}}. \tag{10}$$

For convenience, let $f_s = 1/T_s$. Note that f_s is called the *sampling rate* or *sampling frequency.* Taking the Fourier transform on (10) yields

$$\begin{aligned} G_\delta(f) &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F\left[g(t)e^{j2\pi k f_s t}\right] \\ &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} G(f - kf_s). \end{aligned} \tag{11}$$

The second step is to examine how the signal $g(t)$ may be recovered from the corresponding ideal sampled signal $g_\delta(t)$. Let us make two assumptions.

1. The signal $g(t)$ is a strictly bandlimited signal with bandwidth W Hz (i.e., $G(f) = 0$ for all $|f| > W$).
2. The sampling rate f_s is sufficiently large so that $f_s \geq 2W$.

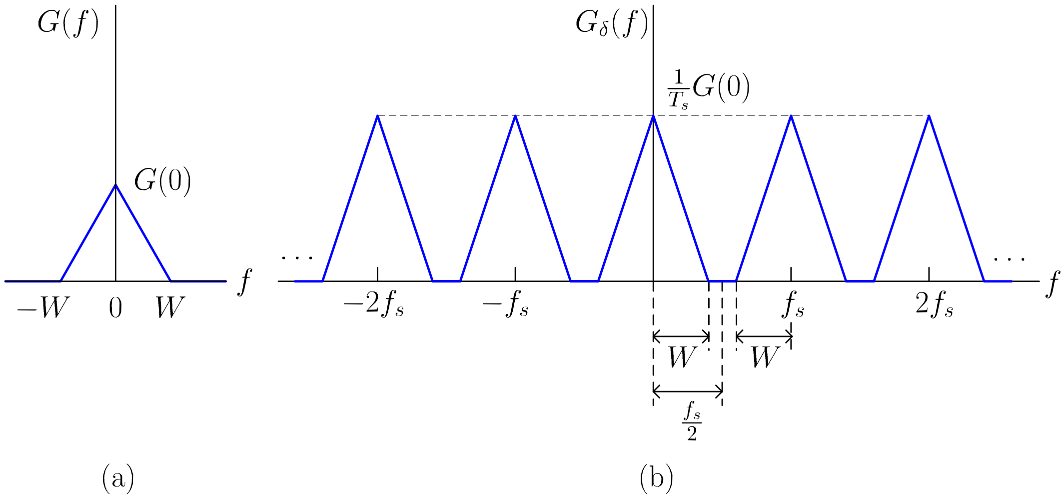


Figure 4: Illustration of the spectrum of $g_\delta(t)$.

Figure 4 illustrates the spectrum $G_\delta(f)$ in (11) under the above two assumptions. We see that each component $G(f - kf_s)$ in (11) is well separated in frequency. In particular, we can retrieve the component $G(f)$ from $G_\delta(f)$ by applying a suitably chosen lowpass filter on $g_\delta(t)$. To be more specific, let

$$H(f) = \begin{cases} 1, & -\frac{f_s}{2} < f < \frac{f_s}{2} \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

be the frequency response of a lowpass filter. We have

$$G_\delta(f)H(f) = \frac{1}{T_s}G(f). \tag{13}$$

This means that if the input of the above lowpass filter is $g_\delta(t)$, then the filter output equals

$$\int_{-\infty}^{\infty} h(\tau)g_\delta(t-\tau)d\tau = \frac{1}{T_s}g(t). \tag{14}$$

In other words, the signal $g(t)$ can be perfectly reconstructed from the ideal sampled signal $g_\delta(t)$ by applying an ideal lowpass filter with bandwidth $f_s/2$. The underlying conditions for the above exact recovery result are the two assumptions mentioned above. Particularly, the assumption $f_s \geq 2W$ is crucial. It also follows that the minimum sampling rate for guarantee of exact recovery is $f_s = 2W$ Hz—this rate is well known as the *Nyquist rate.*

We should additionally describe the so-called interpolation formula of the sampling theorem.

By substituting (4) into (14), we get

$$\begin{aligned} g(t) &= T_s \int_{-\infty}^{\infty} h(\tau) \left[\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-\tau-nT_s) \right] d\tau \\ &= T_s \sum_{n=-\infty}^{\infty} g(nT_s) \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau-nT_s)d\tau \\ &= T_s \sum_{n=-\infty}^{\infty} g(nT_s)h(t-nT_s) \end{aligned} \tag{15}$$

Since the inverse Fourier transform of $H(f)$ in (12) is

$$h(t) = f_s \cdot \mathrm{sinc}(f_s t), \tag{16}$$

Eq. (15) can be further expressed as

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} g(nT_s)\mathrm{sinc}(f_s(t-nT_s)) \\ &= \sum_{n=-\infty}^{\infty} g(nT_s)\mathrm{sinc}\left(\frac{t}{T_s}-n\right). \end{aligned} \tag{17}$$

Eq. (17) is known as the interpolation formula.

3 Spectrum of Pulse-Amplitude Modulated Signals

We now return to the study of PAM. The PAM signal in (1) can be rewritten as

$$s(t) = \int_{-\infty}^{\infty} h(\tau)m_\delta(t-\tau)d\tau, \tag{18}$$

where

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t-nT_s) \tag{19}$$

is the ideal sampled signal of $m(t)$, and $h(t)$ is the pulse shape; note that (18) is obtained by the same way as in (15). Following the same derivations as in the sampling theorem section, the Fourier transform of $s(t)$ is

$$S(f) = H(f)\left[f_s \sum_{k=-\infty}^{\infty} M(f-kf_s) \right] \tag{20}$$

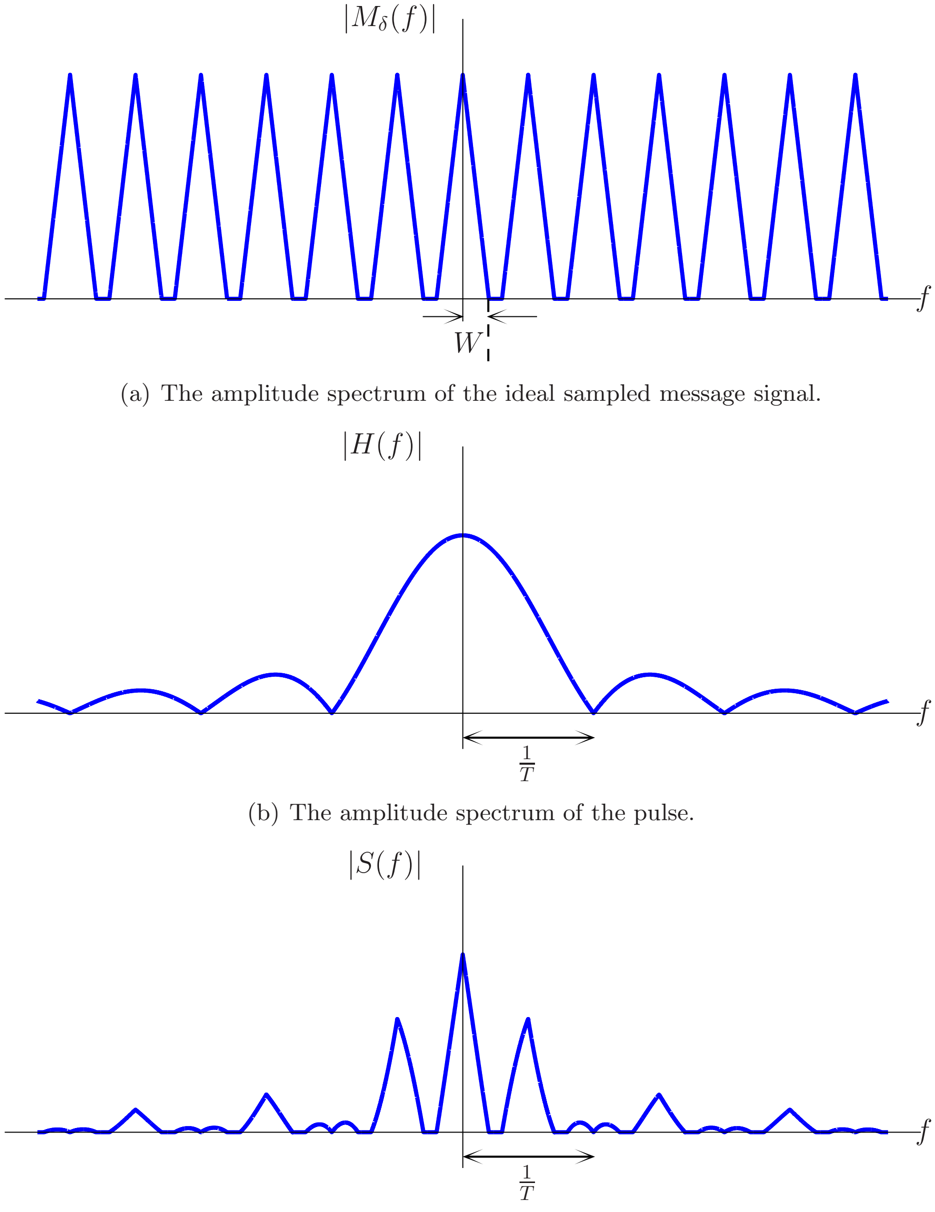
where $f_s = 1/T_s$. Also, for the rectangular pulse in (2), the Fourier transform $H(f)$ is given by

$$H(f) = Te^{-j\pi fT}\mathrm{sinc}(fT). \tag{21}$$

Figure 5 illustrates how the PAM spectrum $S(f)$ may look like. Since $T < T_s$, the PAM bandwidth approximately equals

$$B_T = \frac{1}{T} \text{ Hz},$$

which is measured by using the mainlobe width of $H(f)$. Note that this is a rough bandwidth estimate; from (20) or Figure 5.(c), it is clear that PAM signals are not strictly bandlimited. From the above PAM bandwidth equation, we argue that PAM is not a bandwidth-efficient scheme. To discuss this issue, we assume that the assumptions required for perfect reconstruction in the sampling theorem hold (i.e., $m(t)$ is strictly bandlimited with bandwidth W, and $f_s \geq 2W$). Since $T < T_s$, we have $B_T = \frac{1}{T} \geq \frac{1}{T_s} \geq 2W$—which means that PAM uses at least twice of the message bandwidth to transmit.



(a) The amplitude spectrum of the ideal sampled message signal.

(b) The amplitude spectrum of the pulse.

(c) The amplitude spectrum of the PAM signal

Figure 5: The spectrum of PAM.

The demodulation of PAM signals should also be mentioned. The message signal may be exactly recovered from the PAM signal by first detecting the values of the samples $\{m(nT_s)\}_{n=-\infty}^{\infty}$ from the PAM signal $s(t)$, say, via sampling; and then by applying the reconstruction formula (17) (Of course, it is assumed that the assumptions for perfect reconstruction in the sampling theorem hold). Alternatively, like the sampling theorem, we may also apply a lowpass filter on $s(t)$ to attempt to recover $m(t)$ (but is it exact?).

4 Further Discussion

4.1 Time Sharing

While we have seen that PAM is not a very bandwidth-efficient modulation scheme, its concept of using discrete-time samples to transmit enables another idea—time sharing.

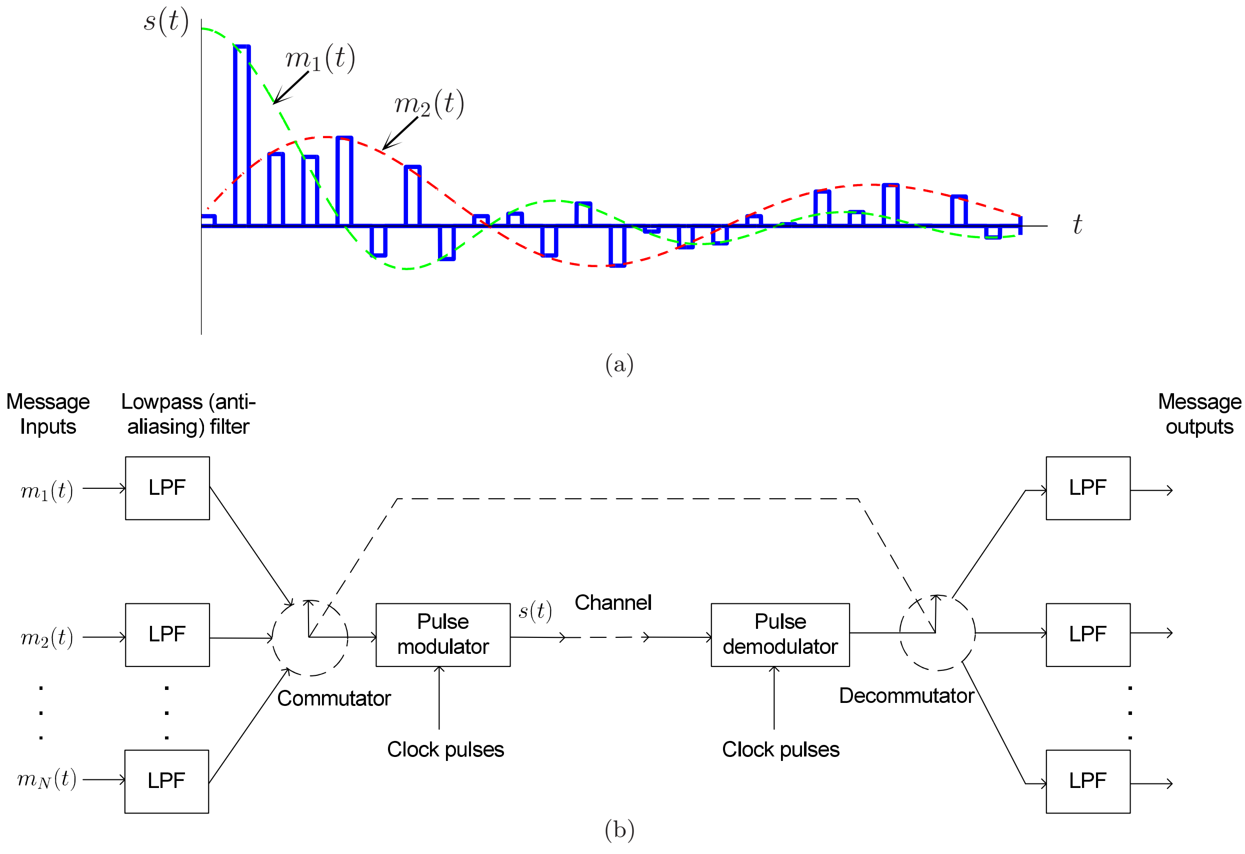


Figure 6: Time division multiplexing. (a) An illustration of the TDM signal for the case of two message signals. (b) An TDM system diagram.

A typical showcasing example for time sharing is the time-division multiplexing (TDM) system. Figure 6.(a) illustrates a TDM waveform for the case of two message signals. As can be seen, the idea to multiplex two (or multiple) message signals within one modulated signal by sharing the channel in a time-interweaving manner. Figure 6.(b) gives a TDM system diagram. What is particularly important here is the *commutator*. The commutator takes N samples, each from one message signal, within the period T_s, and then sequentially interweave them for PAM transmission. At the receiver side, the decommutator does the inverse operation of the commutator to demultiplex the samples. The commutator and decommutator are usually implemented by electronic switching circuitry. Also, the commutator and the decommutator must maintain timing synchronism in order for the system to work properly.

4.2 Impulse Radio

Many modern or advanced wireless communication systems tend to make the transmission bandwidth as efficient as possible. Impulse radio (IR) is a transmission approach that does not follow this philosophy. IR does not have carrier modulation. It directly transmit pulses that are like RF pulses. The pulse width can be ultra-short, talking about nanosecond scales.

IR has been considered in ultra-wideband (UWB) communications. UWB generally refers to an RF band between 3.1 GHz and 10.6 GHz (up to 7.5 GHz in bandwidth). While IR occupies a large portion of spectrum, it has several advantages, e.g., improved capability of penetrating through obstacles. Applications in IR-UWB are usually for low-power, short-range indoor communication. PPM (digital) is one of the modulation schemes in IR.