

## Handout 2: Fourier Transform

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**Suggested Reading:** Chapter 2 of Simon Haykin and Michael Moher, *Communication Systems (5th Edition)*, Wiley & Sons Ltd; or Chapter 3 of B. P. Lathi and Z. Ding, *Modern Digital and Analog Communication Systems (4th Edition)*, Oxford University Press.

The purpose of this handout is to review basic concepts about the Fourier transform, especially those related to communication signals.

## 1 The Fourier Transform

Let  $g(t)$  be a signal in time domain, or, a function of time  $t$ . The Fourier transform of the signal  $g(t)$  is defined as

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft} dt, \quad (1)$$

where  $j = \sqrt{-1}$ , and the variable  $f$  denotes *frequency*. Given a Fourier transform  $G(f)$ , the corresponding signal  $g(t)$  may be obtained by the inverse Fourier transform formula

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft} df. \quad (2)$$

We should emphasize that the time  $t$  is measured in *seconds* (s), and the frequency  $f$  is measured in *Hertz* (Hz). In the literature, it is also common to use *angular frequency* as an alternative way to measure frequency. The angular frequency, denoted herein by  $\omega$ , has the following relationship with the frequency  $f$

$$\omega = 2\pi f,$$

and  $\omega$  is measured in *radians per second* (rad/s). This course will adopt  $f$  (or Hertz) as our main notation for frequency.

In the description above, we have hidden some mathematical details. In particular, from a mathematics viewpoint, a key question is whether the Fourier transform integral (1) exists; the same applies to the inverse Fourier transform integral (2). Simply speaking, the Fourier transform is provably existent for certain classes of signals  $g(t)$ . One such class is that of the finite-energy signals, that is, signals satisfying  $\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$ .

We introduce some shorthand notations for convenience purposes. We use the operators notation

$$G(f) = F[g(t)]$$

to represent the Fourier transform in (1). Likewise,

$$g(t) = F^{-1}[G(f)]$$

represents the inverse Fourier transform in (2). The notation

$$g(t) \rightleftharpoons G(f)$$

means that  $g(t)$  and  $G(f)$  are a Fourier-transform pair (more verbosely, it means that  $g(t)$  is the Fourier transform of  $G(f)$ , and  $G(f)$  the inverse Fourier transform of  $g(t)$ ).

### 1.1 Example: Rectangular Pulse

We consider a rectangular pulse of duration  $T$  and amplitude  $A$ . The following representation is used. Let

$$\text{rect}(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & |t| \geq \frac{1}{2}. \end{cases} \quad (3)$$

We represent our interested pulse signal by

$$g(t) = A \cdot \text{rect}\left(\frac{t}{T}\right). \quad (4)$$

(As a mini-exercise for you, sketch  $\text{rect}(t)$  and  $g(t)$ .) The Fourier transform of  $g(t)$  is given by

$$\begin{aligned} G(f) &= \int_{-T/2}^{T/2} A e^{-j2\pi f t} dt \\ &= A \cdot \frac{\sin(\pi f T)}{\pi f}. \end{aligned} \quad (5)$$

Let us express (5) in a more compact form. Let

$$\text{sinc}(\lambda) = \frac{\sin(\pi \lambda)}{\pi \lambda}$$

denote the sinc function. The Fourier transform of  $g(t)$  in (5) can be equivalently written as

$$G(f) = AT \cdot \text{sinc}(fT). \quad (6)$$

We thus have obtained the Fourier-transform pair

$$A \cdot \text{rect}\left(\frac{t}{T}\right) \Leftrightarrow AT \cdot \text{sinc}(fT) \quad (7)$$

The Fourier transform  $G(f)$  is sketched in Figure 1.

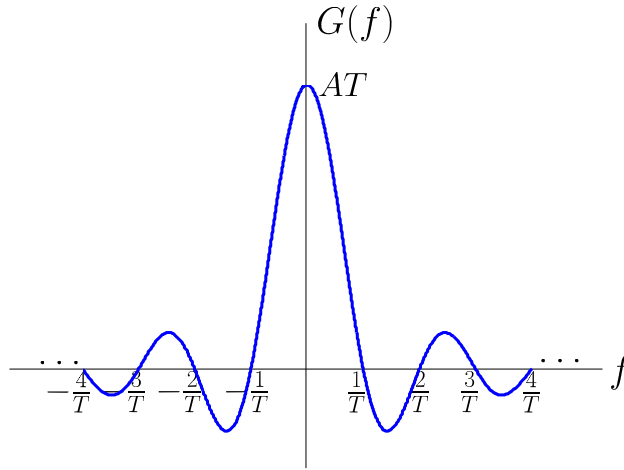


Figure 1: The Fourier transform of a rectangular pulse.

## 1.2 Spectrum

Physically, the Fourier transform of a signal describes the frequency-domain or spectral contents of the signal at various frequencies. In particular, given a signal  $g(t)$ , the corresponding Fourier transform  $G(f)$  can be expressed as

$$G(f) = |G(f)|e^{j\theta(f)}, \quad (8)$$

where  $|G(f)|$  is the absolute value of  $G(f)$  and describes the magnitude of the frequency component of  $g(t)$  at frequency  $f$ ;  $\theta(f)$  is the phase value of  $G(f)$  and describes the phase of the frequency component of  $g(t)$  at frequency  $f$  (note that the Fourier transform  $G(f)$  is generally complex-valued, even though  $g(t)$  is real-valued). We call  $|G(f)|$  the *amplitude spectrum* of  $g(t)$ , and  $\theta(f)$  the *phase spectrum* of  $g(t)$ .

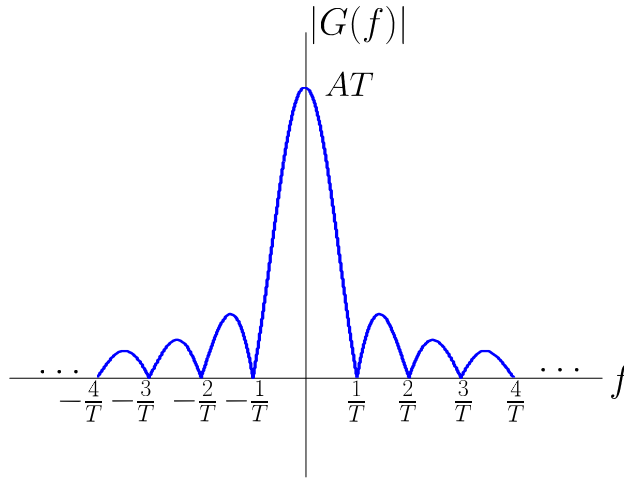


Figure 2: The amplitude spectrum of a rectangular pulse.

Let us take the rectangular pulse example in Section 1.1 as an example. The amplitude spectrum is  $|G(f)| = AT|\text{sinc}(fT)|$ . Figure 2 illustrates the amplitude spectrum. It is observed that the rectangular pulse signal exhibits strong spectral contributions over the frequency interval  $[-\frac{1}{T}, \frac{1}{T}]$ . This further implies that if the pulse signal has a shorter duration  $T$ , then the corresponding Fourier transform will occupy a wider range of frequencies.

## 2 Properties of the Fourier Transform

Given a signal  $g(t)$ , one may obtain the corresponding Fourier transform  $G(f)$  by solving the integral in (1) in a direct manner. Alternatively, we may employ known results or properties of the Fourier transform to derive  $G(f)$ . In fact, an indirect proof based on such properties can sometimes be much simpler than directly tackling the Fourier transform integral. In addition, properties themselves provide insight into analysis of communication signals and systems.

Property	Mathematical Description
1. Linearity	$ag_1(t) + bg_2(t) \Rightarrow aG_1(f) + bG_2(f)$ where $a$ and $b$ are constants
2. Time scaling	$g(at) \Rightarrow \frac{1}{ a }G\left(\frac{f}{a}\right)$ where $a$ is a constant
3. Duality	If $g(t) \Rightarrow G(f)$ , then $G(t) \Rightarrow g(-f)$
4. Time shifting	$g(t - t_0) \Rightarrow G(f)e^{-j2\pi ft_0}$
5. Frequency shifting	$e^{j2\pi f_c t}g(t) \Rightarrow G(f - f_c)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)dt = G(0)$
7. Area under $G(f)$	$g(0) = \int_{-\infty}^{\infty} G(f)df$
8. Differentiation in the time domain	$\frac{d}{dt}g(t) \Rightarrow j2\pi fG(f)$
9. Integration in the time domain	$\int_{-\infty}^t g(\tau)d\tau \Rightarrow \frac{1}{j2\pi f}G(f) + \frac{G(0)}{2}\delta(f)$
10. Conjugate functions	If $g(t) \Rightarrow G(f)$ , then $g^*(t) \Rightarrow G^*(-f)$
11. Multiplication in the time domain	$g_1(t)g_2(t) \Rightarrow \int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda)d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)d\tau \Rightarrow G_1(f)G_2(f)$
13. Rayleigh's energy theorem	$\int_{-\infty}^{\infty}  g(t) ^2 dt = \int_{-\infty}^{\infty}  G(f) ^2 df$

Table 1: Summary of properties of the Fourier transform.

We give a summary of a number of frequently-used Fourier transform properties in Table 1. In the ensuing subsections we give examples on how these properties are used.

## 2.1 The Rectangular Pulse Example Revisited

Consider the rectangular pulse example in Section 1.1 again. An alternative way to solve the Fourier transform is to first solve the Fourier transform of  $\text{rect}(t)$ , which can be easily shown to be

$$\text{rect}(t) \Rightarrow \text{sinc}(f).$$

Then, we use the time scaling property, which says that

$$g(at) \Rightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right),$$

where  $a \neq 0$  is a constant. By the time scaling property, we have

$$A \cdot \text{rect}\left(\frac{t}{T}\right) \Rightarrow AT \cdot \text{sinc}(fT).$$

## 2.2 Example: Rectangular Pulse with Interval $(0, T)$

Consider a rectangular pulse whose time interval is between 0 and  $T$ , viz.

$$g(t) = \begin{cases} A, & 0 < t < T \\ 0, & t \leq 0 \text{ or } t \geq T \end{cases}$$

(again, sketch  $g(t)$ ). The Fourier transform of  $g(t)$  can be obtained by direct integration, and the result is  $G(f) = e^{-j2\pi fT/2} AT \cdot \text{sinc}(fT)$  (try!). Alternatively, we can use the time shifting property—if  $g(t) \Rightarrow G(f)$ , then

$$g(t - t_0) \Rightarrow G(f) e^{-j2\pi f t_0}$$

for any time constant  $t_0$ . To apply the time shifting property, we first rewrite  $g(t)$  as

$$g(t) = A \cdot \text{rect}\left(\frac{t - T/2}{T}\right).$$

We have shown in Section 1.1 or Section 2.1 that  $A \cdot \text{rect}\left(\frac{t}{T}\right) \Rightarrow AT \cdot \text{sinc}(fT)$ . It directly follows from the time shifting property that

$$A \cdot \text{rect}\left(\frac{t - T/2}{T}\right) \Rightarrow e^{-j2\pi fT/2} AT \cdot \text{sinc}(fT)$$

## 2.3 Example: Radio Frequency (RF) Pulse

Consider a pulse signal

$$g(t) = A \cdot \text{rect}\left(\frac{t}{T}\right) \cdot \cos(2\pi f_c t),$$

which is a sinusoidal pulse with frequency  $f_c$ , amplitude  $A$  and duration  $T$  (sketch  $g(t)$  again). The Fourier transform of  $g(t)$  can be solved once again by direct integration, and the result is  $G(f) = \frac{AT}{2} (\text{sinc}((f - f_c)T) + \text{sinc}((f + f_c)T))$ . Here, we solve the Fourier transform by the frequency shifting property: If  $g(t) \Rightarrow G(f)$ , then

$$e^{j2\pi f_c t} g(t) \Rightarrow G(f - f_c)$$

for any real constant  $f_c$ . By noting that

$$\cos(2\pi f_c t) = \frac{1}{2} (e^{j2\pi f_c t} + e^{-j2\pi f_c t})$$

and by applying the frequency shifting property, we get

$$\begin{aligned} F \left[ A \cdot \text{rect} \left( \frac{t}{T} \right) \cdot \cos(2\pi f_c t) \right] &= A \cdot \left\{ F \left[ \text{rect} \left( \frac{t}{T} \right) \cdot e^{j2\pi f_c t} \right] + F \left[ \text{rect} \left( \frac{t}{T} \right) \cdot e^{-j2\pi f_c t} \right] \right\} \\ &= \frac{AT}{2} (\text{sinc}((f - f_c)T) + \text{sinc}((f + f_c)T)). \end{aligned} \quad (9)$$

In this example we are also interested in the amplitude spectrum of the RF pulse. Figure 3 illustrates the amplitude spectrum. In this illustration we assume that  $f_c \gg T$ , so that the two terms in (9) have no significant overlaps. Since the amplitude spectrum is symmetric<sup>1</sup>, let us just look at the positive frequencies ( $f > 0$ ). As can be seen in Figure 3, the significant portion of spectral components of  $g(t)$  lies in the frequency interval  $[f_c - 1/T, f_c + 1/T]$ .

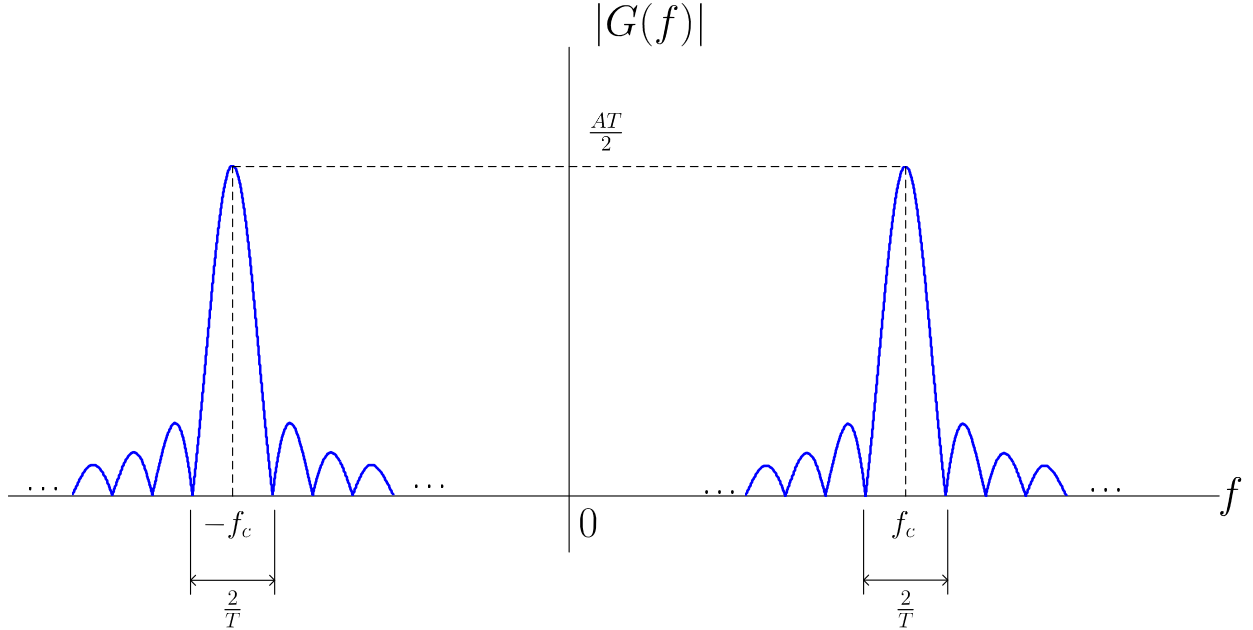


Figure 3: The amplitude spectrum of an RF pulse.

## 2.4 Example: Sinc Pulse

Consider a sinc pulse function

$$g(t) = A \cdot \text{sinc}(2Wt), \quad (10)$$

where  $W \geq 0$  is a constant. We use the duality property to obtain the Fourier transform of  $g(t)$ . The duality property states that if  $g(t) \rightleftharpoons G(f)$ , then

$$G(t) \rightleftharpoons g(-f).$$

<sup>1</sup>It can be proven that for a real-valued  $g(t)$ , the amplitude spectrum  $|G(f)|$  must be symmetric at  $f = 0$ . However, we should note that the same does not apply when  $g(t)$  is a complex-valued function. An example is  $g(t) = Ae^{j2\pi f_c t} \cdot \text{rect}(\frac{t}{T})$ .

In Section 1.1, we have obtained the result  $\text{rect}(t) \Leftrightarrow \text{sinc}(f)$ . Hence, by duality, we have  $\text{sinc}(t) \Leftrightarrow \text{rect}(-f) = \text{rect}(f)$ . By applying time scaling, the following result is yielded

$$A \cdot \text{sinc}(2Wt) \Leftrightarrow \frac{A}{2W} \text{rect}\left(\frac{f}{2W}\right). \quad (11)$$

It is interesting to examine the spectral contents of a sinc pulse. Figure 4 shows the amplitude spectrum of  $g(t) = A \cdot \text{sinc}(2Wt)$ . It is seen that the spectral contents of  $g(t)$  are strictly limited within the frequency interval  $[-W, W]$ . This is unlike rectangular pulses, where spectral “leakage” still appears even when  $f$  approaches infinity or minus infinity; cf. Figure 2.

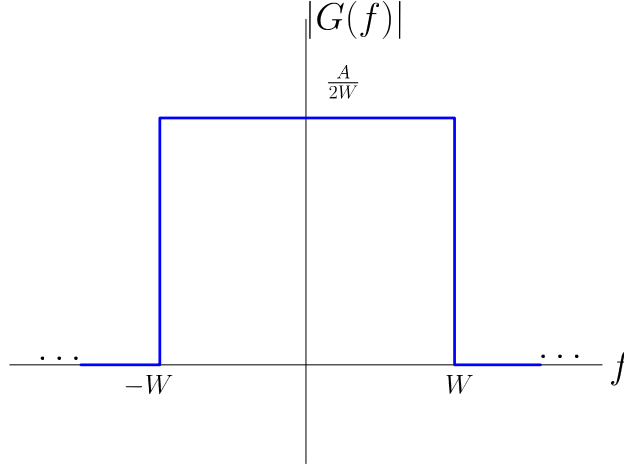


Figure 4: The amplitude spectrum of a sinc pulse.

## 2.5 Other Properties

We should also mention several important properties. The convolution property

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)d\tau \Leftrightarrow G_1(f)G_2(f)$$

will be used when we deal with channels and filters. The property

$$\int_{-\infty}^{\infty} g(t)dt = G(0)$$

is useful in evaluating the mean or “DC” value of a signal. Take the sinc pulse in (10) as an example. While solving the integral  $\int_{-\infty}^{\infty} g(t)dt$  may be difficult, one can obtain the DC value by  $\int_{-\infty}^{\infty} g(t)dt = G(0) = \frac{A}{2W}$  (see (11)). The Rayleigh energy theorem

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df,$$

which physically translates into the implication that the energy evaluated in time domain is identical to the energy evaluated in frequency domain, also proves useful in many occasions. For the sinc

pulse example, we can directly obtain the pulse energy by

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df \\ &= \left( \frac{A}{2W} \right)^2 \int_{-W}^W df \\ &= \frac{A^2}{2W}. \end{aligned}$$

### 3 Signals Through Systems

In many situations we have devices that take a signal as the input and produces another signal as the output. We use the notion of *systems* to characterize such devices. In its generic form, a system is represented by a formula

$$y(t) = T[x(t)],$$

where  $x(t)$  is the input signal,  $y(t)$  is the output signal, and  $T[\cdot]$  denotes a transformation that characterizes the input-output relation of the system. We may also use a diagram, such as the one below, to represent the system.

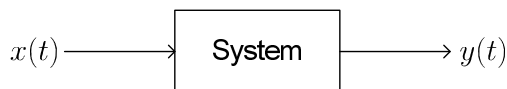


Figure 5: A generic system diagram.

#### 3.1 Linear Time-Invariant Systems

We are interested in a system subclass called the *linear time-invariant (LTI) systems*. You should have learnt the LTI system concepts in ENGG2030 Signals and Systems or in relevant courses. Simply speaking, an LTI system is a system that exhibits linear input-output relationship and does not change with time.

The beauty with LTI systems is that its input-output relationship can always be written as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau, \quad (12)$$

for some time function  $h(t)$  that is called the impulse response of the system. Note that (12) is a convolution integral. Also, by taking the Fourier transform on the both sides of (12), we get

$$Y(f) = H(f)X(f), \quad (13)$$

where  $H(f) = F[h(t)]$  is called the frequency response of the (LTI) system.

#### 3.2 Filters

A filter is a device that is designed to keep frequency components of the input signal that are within a certain frequency region (or a prespecified frequency band), and reject those that are outside the



region. There are many ways to implement a filter in real world, and electronic circuits and DSP technology are two commonly known ways to do so.

Generally, a filter is assumed to be LTI such that (13) applies. Lowpass, highpass and bandpass filters are some of the most frequently seen types of filters in engineering. Figure 6 shows how each type of filters works.

We should note that the illustrations in Figure 6 are ideal. Let us take the lowpass filter as an example. The frequency response of the lowpass filter is an ideal rectangular function

$$H(f) = \begin{cases} 1, & -B \leq f \leq B \\ 0, & |f| > B \end{cases}$$

where  $B$  denotes the cutoff frequency of the lowpass filter. In Section 2.4, we learnt that the corresponding impulse response is

$$h(t) = 2B \cdot \text{sinc}(2Bt).$$

Implementing a filter that has such a sinc-shape impulse response is impossible in reality: it has a very long time spread or duration (see the wave shape in Figure 1), and in fact  $h(t)$  is time unlimited. Having said that, and as in many engineering studies, we can seek good approximations to the ideal lowpass filter that can be effectively realized in real world. In a nutshell, practical filter designs lead to another topic which you can learn later in more advanced courses. In this course where we focus more on communication system architectures, we will assume ideal filtering in many cases so as to understand the concepts more easily.

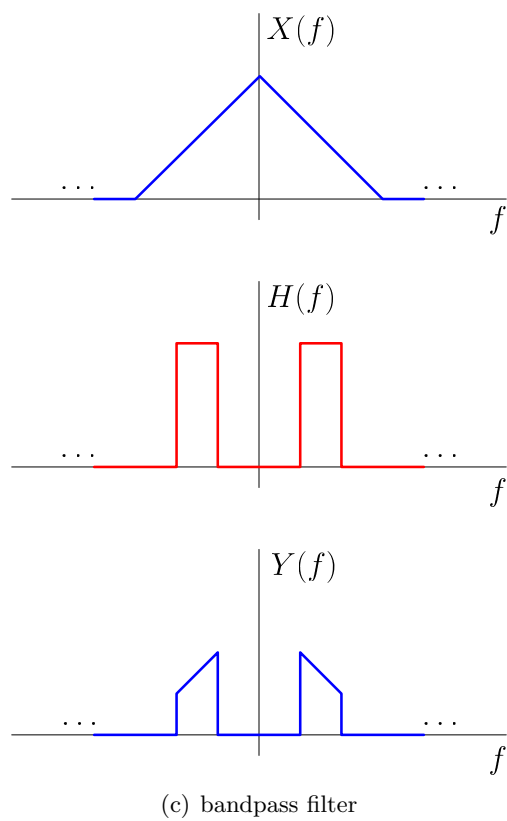
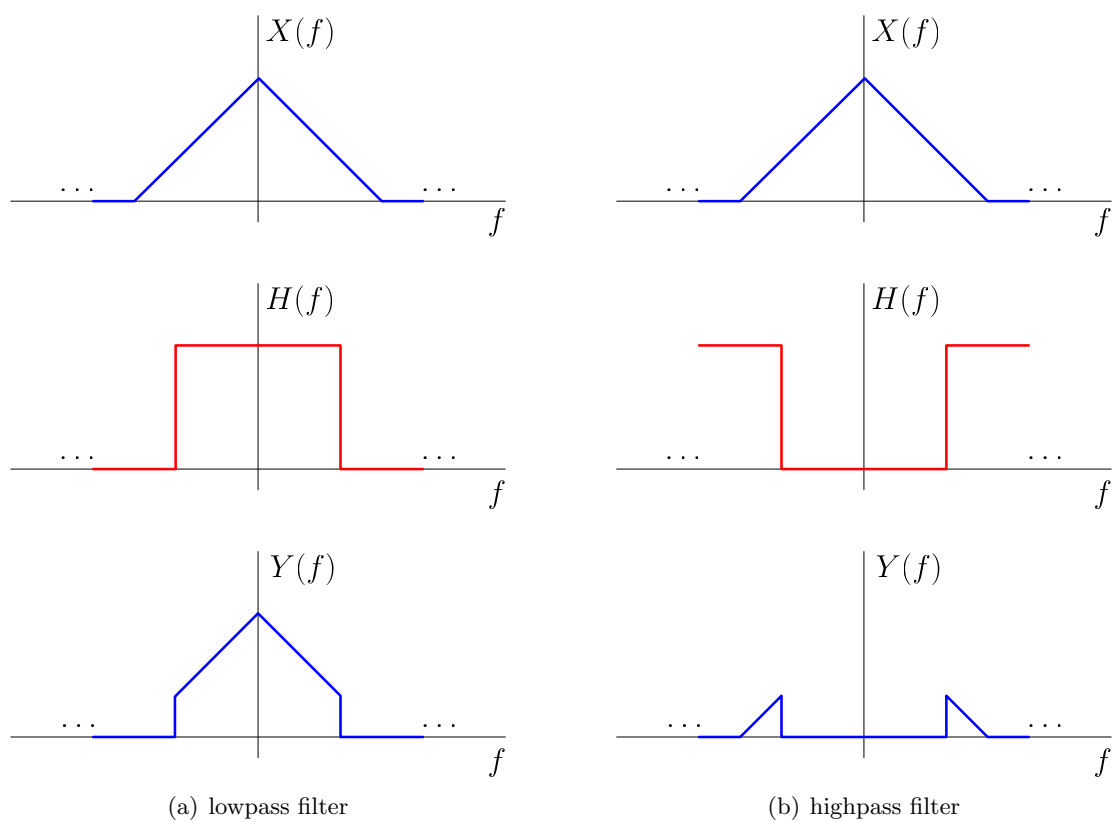


Figure 6: Illustration of filtering.