ELEG 2310B: Principles of Communication Systems	2021-22 First Term

Handout 13: Intersymbol Interference

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Suggested Reading: Chapter 8 of Simon Haykin and Michael Moher, *Communication Systems (5th Edition)*, Wily & Sons Ltd; or Chapter 7 of B. P. Lathi and Z. Ding, *Modern Digital and Analog Communication Systems (4th Edition)*, Oxford University Press.

This handout concentrates on the issue of intersymbol interference (ISI) arising in digital baseband transmission over a non-ideal channel.

1 Formulation

1.1 System Model

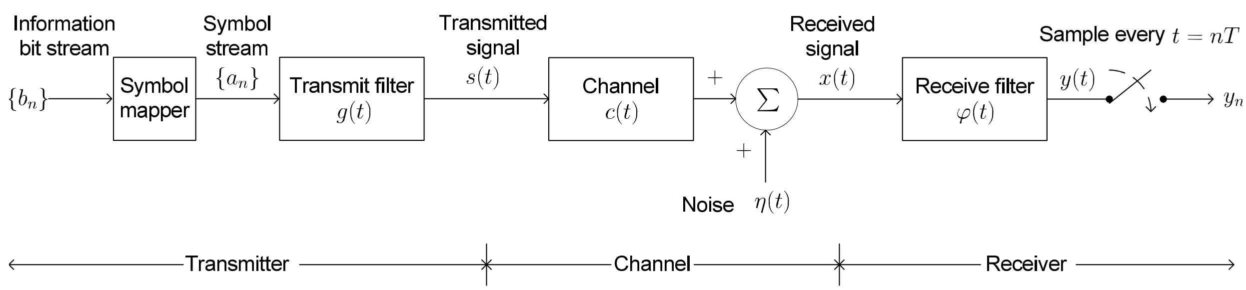

Figure 1: A system diagram for the overall baseband PAM system.

In Handout 11, we studied 2-ary PAM transmission in the presence of channel distortions and noise. Figure 1 shows the whole system in one diagram. A summary of the signal model is as follows. The transmitted PAM signal is given by

$$s(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT), \tag{1}$$

where $\{a_n\}$ is the symbol stream, T is the symbol interval, and $g(t)$ is the transmitted pulse shape. The received signal is modeled as

$$x(t) = \int_{-\infty}^{\infty} c(\tau)s(t-\tau)d\tau + \eta(t), \tag{2}$$

where $c(t)$ is the channel impulse response, and $\eta(t)$ is noise. Recall that $c(t)$ models characteristics of the physical transmission medium, and in this handout we assume that $c(t)$ is non-ideal in the sense that $c(t)$ may introduce distortions to the transmitted pulses. The receiver side is reminiscent of the matched filter described in Handout 11 (with some minor differences). The received signal $x(t)$ first passes through a receive filter $\varphi(t)$. The signal after receive filtering is

$$y(t) = \int_{-\infty}^{\infty} \varphi(\tau) x(t - \tau) d\tau. \tag{3}$$

Then, sampling is performed to obtain signal samples

$$y_n = y(nT). \tag{4}$$

The rationale of (3)–(4) is to perform the correlation receiver operation, which we claim it to be a "good" way to deal with noise in Handout 11. The sample sequence $\{y_n\}$ is used to perform detection of $\{a_n\}$.

Let us quickly review the ideal case where the received signal $x(t)$ is simply $x(t) = s(t) + \eta(t)$ (i.e., no channel distortion) and the transmitted pulses do not overlap in time. Following the matched filter principle described in Handout 11, we can show that the receive filter should be chosen as

$$\varphi(t) = g(-t) \tag{5}$$

(note that there is some subtle difference compared with the matched filter in Handout 11; be careful). Subsequently, it can be proven that the signal samples y_n equal $y_n = E_g \cdot a_n + v_n$, where E_g is pulse energy and v_n is a noise term. We may then detect a_n by applying decision on y_n.

Now, consider the non-ideal case where the transmitted pulses may be distorted by the channel. For convenience, $\star$ denote the convolution operator; i.e., $a(t) \star b(t)$ means that $a(t) \star b(t) = \int_{-\infty}^{\infty} a(\tau) b(t - \tau) d\tau$. As will be derived in detail in Section 1.3, the signal after receive filtering can be expressed as

$$\boxed{y(t) = \sum_{k=-\infty}^{\infty} a_k h(t - kT) + v(t),} \tag{6}$$

where

$$v(t) = \int_{-\infty}^{\infty} \varphi(\tau) \eta(t - \tau) d\tau \tag{7}$$

is noise after receive filtering, and

$$h(t) = g(t) \star c(t) \star \varphi(t). \tag{8}$$

is the pulse shape after passing through the channel and receive filter. We observe from (6) that if we ignore the noise $v(t)$, then the signal after receive filtering is just a PAM waveform with a "new" pulse shape $h(t)$ (cf. the PAM signal expression in (1)).

1.2 Intersymbol Interference

Let us examine the received signal sample sequence $\{y_n\}$ under the model in (6). We have

$$y_n = y(nT) = \sum_{k=-\infty}^{\infty} a_k h(nT - kT) + v(nT). \tag{9}$$

For notational convenience, let $h_n = h(nT)$ and $v_n = v(nT)$. The above equation can be rewritten as

$$\boxed{y_n = \sum_{k=-\infty}^{\infty} a_k h_{n-k} + v_n.} \tag{10}$$

As can be seen above, the sample sequence $\{y_n\}$ is a discrete-time convolution of $\{h_n\}$ and $\{a_n\}$, plus noise. With the discrete-time received signal model in (10), we can define more precisely what is ISI. Eq. (10) can be expressed as

$$y_n = a_n h_0 + \underbrace{\sum_{k=-\infty, k\neq n}^{\infty} a_k h_{n-k}}_{\text{ISI}} + v_n. \tag{11}$$

The second term in the above equation is defined as *intersymbol interference*; it is interference from other symbols transmitted at different intervals. ISI, if not handled properly, can result in significant performance degradation.

1.3 Proof of (6)

Before proceeding further, we should describe how the signal model (6) is derived. The proof is as follows. We substitute (1) into (2) to get

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} c(\tau) \left[\sum_{k=-\infty}^{\infty} a_k g(t - \tau - kT) \right] d\tau + \eta(t), \\ &= \sum_{k=-\infty}^{\infty} a_k \left[\int_{-\infty}^{\infty} c(\tau) g(t - kT - \tau) d\tau \right] + \eta(t). \end{aligned} \tag{12}$$

Let $q(t) = c(t) \star g(t)$. Eq. (12) can be more compactly represented by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k q(t - kT) + \eta(t). \tag{13}$$

Substituting (13) into (3) yields

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \varphi(\tau) \left[\sum_{k=-\infty}^{\infty} a_k q(t - \tau - kT) \right] d\tau + \underbrace{\int_{-\infty}^{\infty} \varphi(\tau) \eta(t - \tau) d\tau}_{=v(t)} \\ &= \sum_{k=-\infty}^{\infty} a_k \left[\int_{-\infty}^{\infty} \varphi(\tau) q(t - kT - \tau) d\tau \right] + v(t) \\ &= \sum_{k=-\infty}^{\infty} a_k [\varphi(t - kT) \star q(t - kT)] + v(t). \end{aligned} \tag{14}$$

By noting that $\varphi(t) \star q(t) = \varphi(t) \star g(t) \star c(t) = h(t)$, the desired result in (6) is obtained.

2 Pulse Shaping

Naturally, we wish to be free from ISI effects[1]. We say that the *zero-ISI condition* is satisfied if

$$h_0 = h(0) \neq 0, \quad \text{and} \quad h_n = h(nT) = 0 \text{ for all integer } n \neq 0. \tag{15}$$

A possible way to achieve zero ISI is to consider *pulse shaping*. Specifically, the idea is to design the transmitted pulse shape $g(t)$, as well as the receive filter $\varphi(t)$, such that the composite pulse shape $h(t)$ satisfies (15).

2.1 Ideal Lowpass Channel

Pulse shaping may be best understood by studying the special case of ideal lowpass channels—and our study will mainly focus on the ideal lowpass channel case. We say a channel $c(t)$ to be *ideal lowpass with bandwidth* B_C if its frequency response satisfies

$$C(f) = A_c e^{-j2\pi f t_c}, \text{ for all } |f| \leq B_C, \tag{16}$$

where $A_c > 0$ and $t_c \geq 0$ are some constants. Physically, an ideal lowpass channel does not introduce distortions to input signals whose spectral content is strictly limited within $[-B_C, B_C]$. Specifically, for a strictly bandlimited pulse shape $g(t)$ with bandwidth B_C (or equivalently, $G(f) = 0$ for all $|f| > B_C$), we have

$$g(t) \star c(t) = F^{-1}[G(f)C(f)] = F^{-1}[A_c G(f) e^{-j2\pi f t_c}] = A_c \cdot g(t - t_c), \tag{17}$$

where the pulse after passing through channel, $g(t) \star c(t)$, is nothing more than an amplitude-scaled and time-delayed version of the original pulse $g(t)$. An example of an approximately ideal lowpass channel is given as follows.

Example 1 Consider a channel with an exponentially decaying impulse response

$$c(t) = \begin{cases} ae^{-at}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

for some constant $a > 0$. The Fourier transform of $c(t)$ is known to be

$$C(f) = \frac{1}{1 + \frac{j2\pi f}{a}}. \tag{19}$$

We see from the above equation that

$$C(f) \approx 1, \quad \text{for } f \text{ such that } \left| \frac{j2\pi f}{a} \right| \ll 1. \tag{20}$$

Hence, we may approximate $c(t)$ as an ideal lowpass channel if we choose B_C as a number much smaller than $\frac{|a|}{2\pi}$.

[1]As an aside, ISI is not necessarily undesirable. Some advanced communication studies suggest that if we can manage interference well, then we may achieve better system capacity than that of enforcing zero interference.

2.2 Zero-ISI Pulse Shaping

Now we describe the pulse shaping problem. The following assumptions are made:

- the channel $c(t)$ is ideal lowpass with bandwidth B_C; and
- the transmitted pulse shape $g(t)$ is strictly bandlimited with bandwidth no greater than B_C.

Then, by (17), we have

$$g(t) \star c(t) = A_c \cdot g(t - t_c), \tag{21}$$

For simplicity, let us further assume $A_c = 1$ and $t_c = 0$ so that

$$g(t) \star c(t) = g(t) \tag{22}$$

(note that we can choose not to apply the above simplifying assumption, although the corresponding results would become a bit more tedious than the derivations to be shown). Moreover, following the discussion in Section 1.1, we choose the receive filter as $\varphi(t) = g(-t)$. Applying the above choice of $\varphi(t)$ to (8) and (22), the composite pulse shape becomes

$$h(t) = g(t) \star g(-t), \tag{23}$$

whose Fourier transform is

$$H(f) = G(f) \cdot G^*(f) = |G(f)|^2. \tag{24}$$

The pulse shaping problem can be divided into two tasks, namely,

i) finding $h(t)$ such that the zero-ISI condition in (15) holds and $h(t)$ is strictly bandlimited within B_C; and

ii) finding $g(t)$ such that, given the pulse shape $h(t)$ obtained in Task i), $g(t)$ satisfies (23) or (24).

The pulse shaping problem posed above is, in a nutshell, not a trivial subject. However, there are straightforward examples and standard solutions. A simple example we should know about is described below.

Example 2 Consider the sinc pulse

$$h(t) = \text{sinc}\left(\frac{t}{T}\right). \tag{25}$$

As can be observed in Figure 2(a), the above sinc pulse satisfies the zero-ISI condition. Since the Fourier transform of $h(t)$ is

$$H(f) = T \cdot \text{rect}(fT), \tag{26}$$

the pulse $h(t)$ is strictly bandlimited with bandwidth $\frac{1}{2T}$; see Figure 2(b). Hence, Task i) can be accomplished by using a sinc pulse, as far as $\frac{1}{2T} \leq B_C$. To achieve Task ii), we notice that if we choose

$$G(f) = \sqrt{H(f)}, \tag{27}$$

then (24) holds. For the sinc pulse choice here, (27) simply equals

$$G(f) = \sqrt{T} \cdot \text{rect}(fT) \tag{28}$$

and the corresponding pulse in time domain is

$$g(t) = \frac{1}{\sqrt{T}} \mathrm{sinc}\left(\frac{t}{T}\right). \tag{29}$$

It is interesting to note that the resultant transmitted PAM signal has pulses overlapping in time. Figure 2(c) gives an illustration.

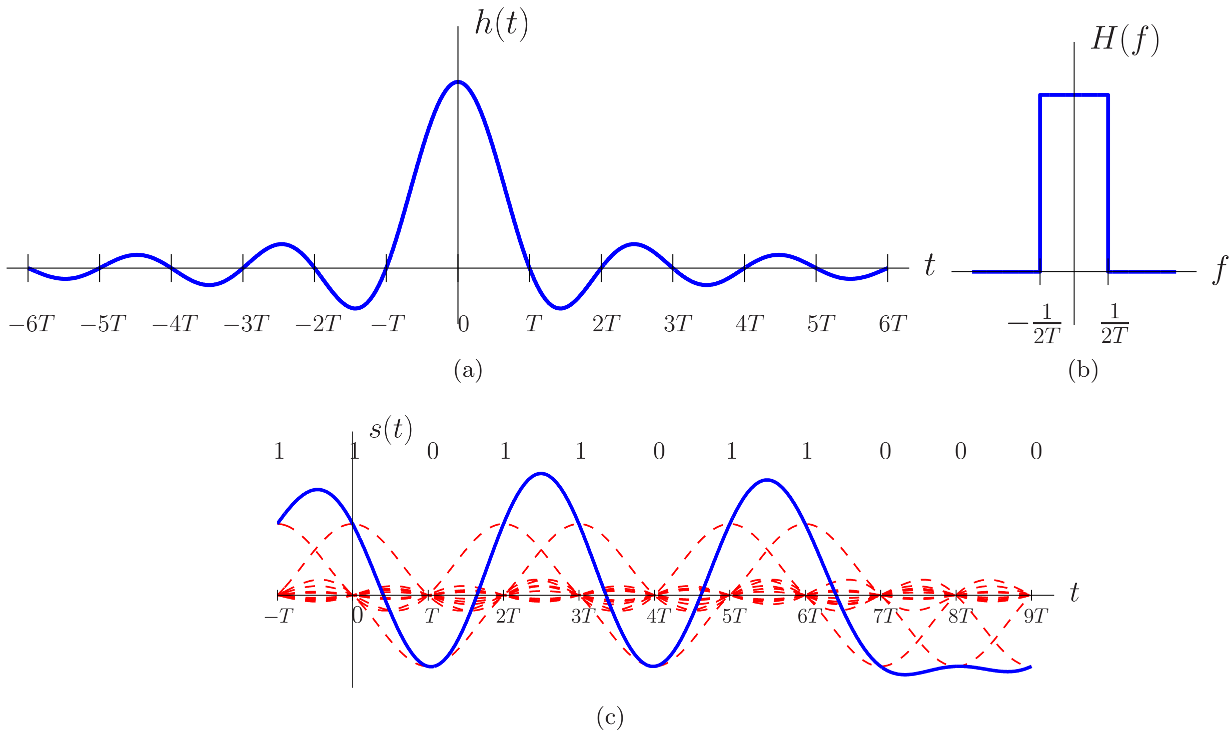

Figure 2: Pulse shaping via the sinc function. (a) The pulse shape. (b) The corresponding spectrum. (c) An illustration of the corresponding PAM waveform. In Figure 2(c), the red dashed lines show the transmitted pulses while the blue solid line shows the resultant transmitted PAM signal.

While we have seen in the above example that the pulse shaping problem can be easily solved by a sinc pulse, the sinc pulse is not a popular pulse shaping solution in real-world digital communication systems. There are other practical issues, such as sensitivity to timing errors and implementation efficiency, that should also be taken into account when designing a pulse shape. These issues are somehow beyond the scope of this handout, and please read the textbooks for details. Before we finish, we should briefly mention a popularized and rather standard class of pulse shapes, namely, the *raised cosine pulse shapes*. A raised cosine pulse shape takes the form

$$h(t) = \mathrm{sinc}\left(\frac{t}{T}\right) \cdot \frac{\cos(\pi \alpha t/T)}{1 - 4\alpha^2 t^2/T^2} \tag{30}$$

where $0 \leq \alpha \leq 1$ is called the *rolloff factor*. It can be seen from the sinc term of (30) that raised cosine pulses satisfy the zero-ISI condition. Figure 3 illustrates the raised cosine pulses under various values of α. The corresponding Fourier transform is shown to be

$$H(f) = \begin{cases} T, & 0 \leq |f| \leq \dfrac{1-\alpha}{2T} \\ \dfrac{T}{2}\left\{1 + \cos\left[\dfrac{\pi T}{\alpha}\left(|f| - \dfrac{1-\alpha}{2T}\right)\right]\right\}, & \dfrac{1-\alpha}{2T} < |f| \leq \dfrac{1+\alpha}{2T}, \\ 0, & |f| > \dfrac{1+\alpha}{2T} \end{cases} \tag{31}$$

According to (31), the bandwidth of $h(t)$ is $\dfrac{1+\alpha}{2T}$.

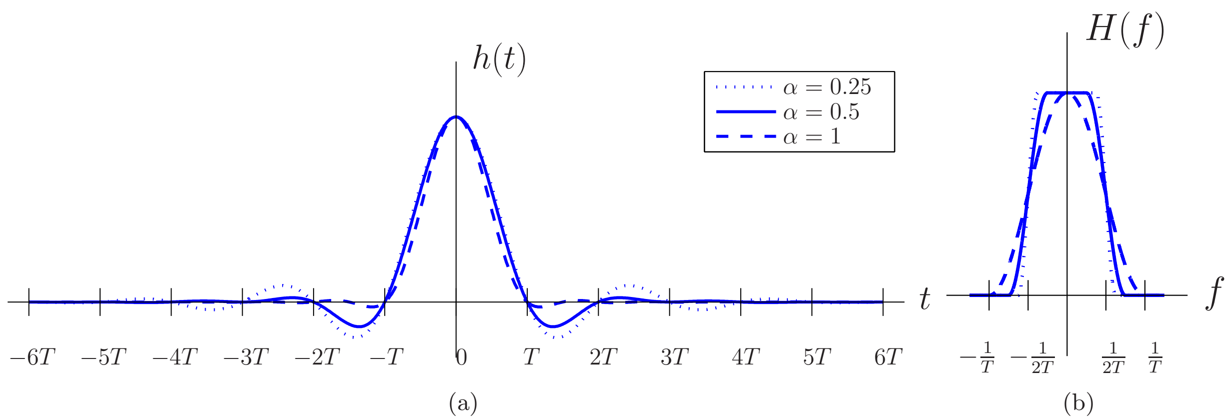

Figure 3: Pulse shaping via the raised cosine function. (a) The pulse shapes. (b) The corresponding spectra.

3 Further Discussion

We still have not discussed the spectrum and transmission bandwidth of digital PAM signals. Binary data are often random in nature, and the amplitude spectrum concepts we have frequently used can no longer be applied to digital communication signals. The treatment of spectra for digital (and random) PAM signals requires the concepts of *power spectral density* in random processes, which is not covered in this course. Simply speaking, power spectral density is like the amplitude spectrum we previously used, except that power spectral density is for random signals. Under some mild assumption, the power spectral density of the 2-ary PAM signal $s(t)$ is known to be

$$\mathcal{S}_s(f) = \frac{1}{T}|G(f)|^2. \tag{32}$$

Hence, the transmission bandwidth is determined by the bandwidth of the pulse shape.

Also, it should be noted that the pulse shaping technique studied above is just among one of the many ways to tackle ISI. An alternative solution is to employ *equalization*, where a digital filter is applied to the sample sequence $\{y_n\}$ to mitigate or neutralize ISI. Another is *orthogonal frequency division multiplexing (OFDM)*, where a different modulation scheme is employed to achieve zero ISI (without the ideal lowpass channel assumption).