ELEG 2310B: Principles of Communication Systems **2021-22 First Term**

Handout 12: Error Probability Analysis of Binary Detection

Instructor: Wing-Kin Ma

Suggested Reading: Chapter 8 of Simon Haykin and Michael Moher, *Communication Systems (5th Edition)*, Wily & Sons Ltd; or Chapter 13 of B. P. Lathi and Z. Ding, *Modern Digital and Analog Communication Systems (4th Edition)*, Oxford University Press.

In the previous handout, we are faced with a binary detection problem. Specifically, in Section 3 of the last Handout, "Detection in Noise," a concept called the correlation receiver was introduced for detection of 2-ary PAM in the presence of noise. There, the correlation receiver first yields a signal sample

$$y_0 = a_0 \cdot E_g + v_0 \tag{1}$$

where a_0 is the transmitted (or true) symbol at the 0th symbol interval, E_g is pulse energy and v_0 is noise. Then, the receiver detects a_0 by the decision rule

$$\hat{a}_0 = \begin{cases} -1, & \text{if } y_0 \leq 0 \\ 1, & \text{if } y_0 > 0 \end{cases} \tag{2}$$

Now, the question arising is how frequent or unfrequent the correlation receiver makes an incorrect decision in the presence of noise. This handout addresses this question by probabilistic performance analysis.

Note that in this particular handout, probability concepts and properties are extensively used. Do not hesitate to ask if you have any question.

1 Formulation

We use random variables and apply probability models to characterize (1). The formulation is described as follows. Let y be a random variable, which characterizes y_0 in (1). The signal model for y is formulated as

$$y = \begin{cases} -\rho + \nu, & a = -1 \\ \rho + \nu, & a = 1 \end{cases} \tag{3}$$

where $\rho > 0$ is a constant, ν is a random variable characterizing the noise term v_0 in (1), and a is a discrete random variable taking the value of either -1 and 1. Moreover, we make the following two probabilistic assumptions

1. a is equiprobable; i.e, $\Pr(a = -1) = \Pr(a = 1) = \frac{1}{2}$.
2. ν follows a Gaussian distribution with mean zero and variance σ^2; i.e., the probability density function of ν is given by

$$p_\nu(v) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2} v^2}. \tag{4}$$

The first assumption seems reasonable since the distribution of "0" and "1" in binary data should usually be quite uniform (like tossing a coin). The second assumption, which assumes noise to be Gaussian, may not be as obvious to you. Simply stated, many of noises encountered in communications are found to be Gaussian, or can be well approximated as Gaussian.

Let $\hat{a}$ be a decision of the random variable a based on the observation y. The decision rule is to decide $\hat{a} = 1$ if $y > 0$, and $\hat{a} = 1$ if $y \leq 0$—the same as (2). Our task is to solve the *error probability*

$$\Pr(\hat{a} \neq a).$$

To be specific, the probability $\Pr(\hat{a} \neq a)$ is the symbol error probability as well as the bit error probability.

Before we proceed, we should note that the variance σ^2 physically represents the noise power. Thus, the quantity

$$\frac{\rho^2}{\sigma^2}$$

describes the signal-to-noise ratio (SNR) of the model in (3).

2 Error Probability Analysis

We divide the error probability analysis into three steps.

Step 1) Consider the hypothesis $a = -1$. We aim at deriving the conditional probability

$$\Pr(\hat{a} = 1 \mid a = -1);$$

i.e., the probability of making the decision $\hat{a} = 1$, given the event that the true symbol is $a = -1$. The expression is as follows:

$$\begin{aligned}
\Pr(\hat{a} = 1 \mid a = -1) &= \Pr(y > 0 \mid a = -1) && \text{(5a)}\\
&= \Pr(-\rho + \nu > 0) && \text{(5b)}\\
&= \Pr(\nu > \rho) && \text{(5c)}\\
&= \int_{\rho}^{\infty} p_{\nu}(v) dv && \text{(5d)}\\
&= \int_{\rho}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2\sigma^2} v^2} dv. && \text{(5e)}
\end{aligned}$$

Note that (5b) is obtained by applying the uppercase of (3). By a change of variable $z = v/\sigma$, we can simplify (5e) to

$$\Pr(\hat{a} = 1 \mid a = -1) = \int_{\rho/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2} z^2} dz. \tag{6}$$

Unfortunately, the integral at the right-hand side of (6) does not admit a simple expression. However, it can be numerically computed. For convenience, let

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2} z^2} dz. \tag{7}$$

Note that $Q(x)$ is known as the *Q-function*. The Q-function is frequently encountered in digital communications, and software like MATLAB has specific functions for evaluating the numerical values of the Q-function. Eq. (6) can be written as

$$\Pr(\hat{a} = 1 \mid a = -1) = Q\left(\frac{\rho}{\sigma}\right). \tag{8}$$

Step 2) Consider the hypothesis $a = 1$. Following the same spirit as in Step 1, we examine the conditional probability $\Pr(\hat{a} = -1 \mid a = 1)$:

$$\begin{aligned}
\Pr(\hat{a} = -1 \mid a = 1) &= \Pr(y \leq 0 \mid a = 1) && \text{(9a)}\\
&= \Pr(\rho + \nu \leq 0) && \text{(9b)}\\
&= \Pr(\nu \leq -\rho) && \text{(9c)}\\
&= \int_{-\infty}^{-\rho} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2\sigma^2}v^2} dv. && \text{(9d)}
\end{aligned}$$

By a change of variable $w = -v/\sigma$, Eq. (9d) is re-expressed as

$$\Pr(\hat{a} = -1 \mid a = 1) = \int_{\rho/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}w^2} dw = Q\left(\frac{\rho}{\sigma}\right). \tag{10}$$

Step 3) We are now ready to deal with the error probability $\Pr(\hat{a} \neq a)$.

$$\begin{aligned}
\Pr(\hat{a} \neq a) &= \Pr(\hat{a} \neq a, a = -1) + \Pr(\hat{a} \neq a, a = 1) && \text{(11a)}\\
&= \Pr(\hat{a} = 1, a = -1) + \Pr(\hat{a} = -1, a = 1) && \text{(11b)}\\
&= \Pr(\hat{a} = 1 \mid a = -1)\Pr(a = -1) + \Pr(\hat{a} = -1 \mid a = 1)\Pr(a = 1) && \text{(11c)}\\
&= Q\left(\frac{\rho}{\sigma}\right) \cdot \frac{1}{2} + Q\left(\frac{\rho}{\sigma}\right) \cdot \frac{1}{2} && \text{(11d)}\\
&= Q\left(\frac{\rho}{\sigma}\right), && \text{(11e)}
\end{aligned}$$

where (11c) is due to Bayes' rule, and (11d) is obtained by substituting (8), (10) and $\Pr(a = -1) = \Pr(a = 1) = 1/2$ into (11c).

To conclude, the error probability is

$$\boxed{\Pr(\hat{a} \neq a) = Q\left(\frac{\rho}{\sigma}\right).} \tag{12}$$

Figure 1 shows the (numerically computed) error probability with respect to the SNR ρ^2/σ^2. It can be seen that the error probability reduces very significant with the SNR.

3 Further Analysis and Implication

(Note that this is an advanced topic) While we can evaluate the error probability by numerically computing the Q-function, we also want to analyze the error probability. In particular, can we prove in what way the error probability reduces with the SNR ρ^2/σ^2?

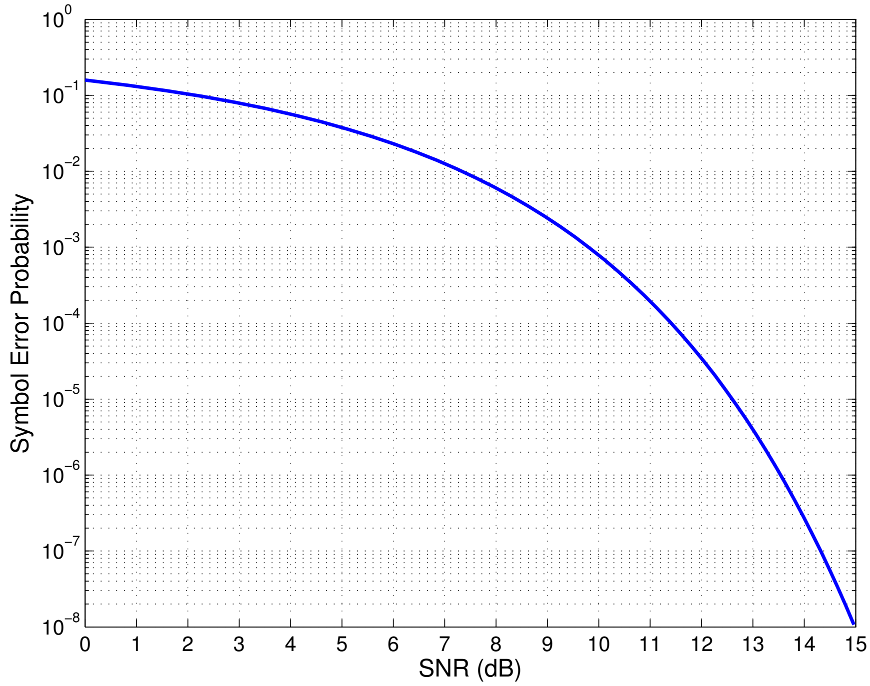

Figure 1: Error probability versus the SNR.

We answer the above question by proving an upper bound on the error probability. Let us consider the Q-function in (7). Let $u(z)$ be the unit step function, i.e., $u(z) = 1$ if $z \geq 0$, and $u(z) = 0$ otherwise. The Q-function can be equivalently expressed as

$$Q(x) = \int_{-\infty}^{\infty} u(z-x) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2} dz. \tag{13}$$

The key trick we apply is that for any $\alpha \geq 0$, it holds true that

$$u(z) \leq \mathrm{e}^{\alpha z} \text{ for any } z. \tag{14}$$

Substituting (14) into (13) yields

$$Q(x) \leq \int_{-\infty}^{\infty} \mathrm{e}^{\alpha(z-x)} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2} dz \tag{15a}$$

$$= \mathrm{e}^{-\alpha x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\alpha z} \cdot \mathrm{e}^{-\frac{1}{2}z^2} dz \tag{15b}$$

$$= \mathrm{e}^{-\alpha x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{\alpha^2}{2}} \cdot \mathrm{e}^{-\frac{(z-\alpha)^2}{2}} dz \tag{15c}$$

$$= \mathrm{e}^{-\alpha x} \cdot \mathrm{e}^{\frac{\alpha^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(z-\alpha)^2}{2}} dz}_{=1} \tag{15d}$$

$$= \mathrm{e}^{-\frac{x^2}{2} + \frac{(\alpha-x)^2}{2}}. \tag{15e}$$

where the integral in (15d) integrates the area under a Gaussian probability density function (with mean α and variance 1), which is one. Next, we wish to choose $\alpha \geq 0$ such that (15e) is the smallest. It is seen that if $x \geq 0$, then $\alpha = x$ leads to the smallest value of (15e). Hence, we obtain the following upper bound for the Q-function:

$$\boxed{Q(x) \leq \mathrm{e}^{-\frac{x^2}{2}}, \quad \text{for any } x \geq 0.} \tag{16}$$

The above upper bound is known as the *Chernoff bound* of the Q-function.

Now, let us apply the Chernoff bound to the error probability:

$$\Pr(\hat{a} \neq a) \leq e^{-\frac{1}{2} \cdot \frac{\rho^2}{\sigma^2}} \tag{17}$$

We see something appealing from the above equation: *The error probability decreases at least in an exponential manner with respect to the SNR*—which is very good!