Foundametnal Course on
Probability, Random Variable and Random Processes

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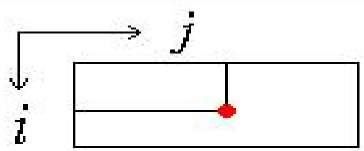
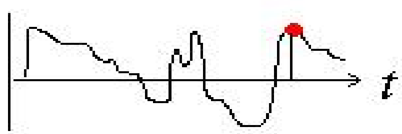
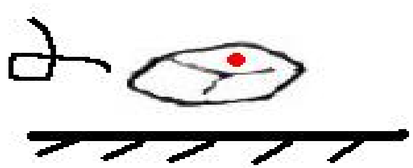
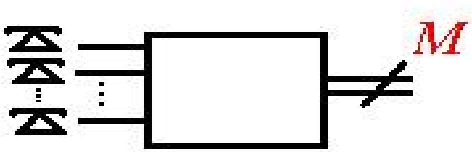
- Probability Theory
- Random Variables
- Random Processes
  - Stationary RP & Ergodic RP
  - Gaussian RP
  - Filtering of RP

Random Variables (2)

The outcomes of the experiments below are random variables whose values are defined at each sample points $s_1, s_2, \ldots$ in a sample space S

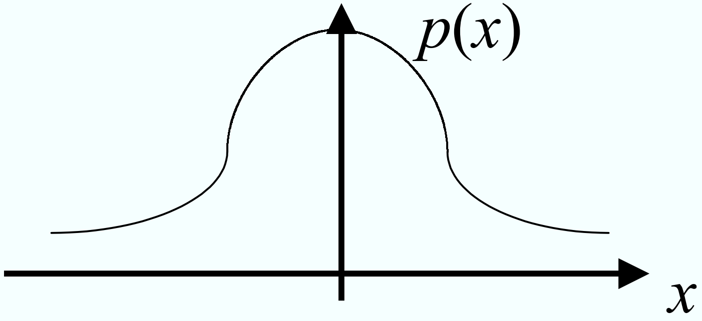
Sample space is the set corresponds to all possible outcomes of an experiment.

A sample point is a point at which an outcome of an experiment is sampled.

experiment	outcome	sample space	sample point
	luminance $x(i,j)$	$\in [0, 255]$	discrete (i,j)
	voltage $x(t)$	$\in (-\infty, \infty)$	continuous t
	$x(i)$	$\in [1,6]$	discrete i
	$k(t)$ lines busy	$\in [0,M]$	continuous t

Definition of Probability Density Function $p.d.f.$ (3)

Def: Let x represent a continuous random variable in a sample space S. For each x we have a $p.d.f.$ $p(x)$ which is a function that satisfies the following:



$$1)\ p(x) \geq 0\ \forall x \in S$$

$$2)\ \int_S p(x)\,dx = 1$$

$3)\ \forall x_1 < x_2$ in S

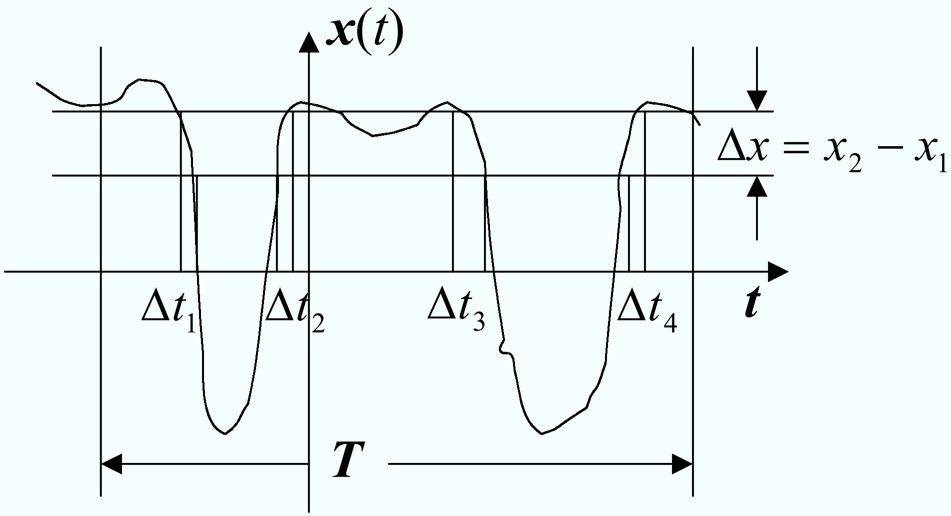
The probability of $x \in [x_1, x_2] = P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} p(x)\,dx$

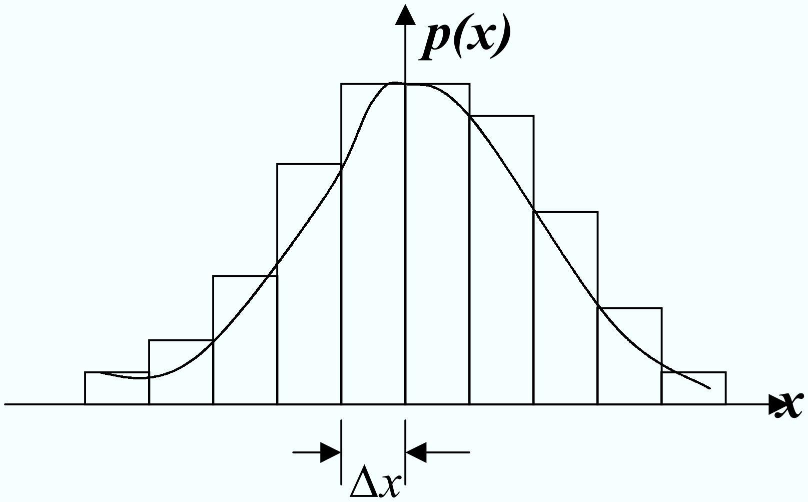
If x is continuous at a, then the probability of $x=a$ $= P(x=a) = \int_a^a p(x)\,da = 0$

The 3 axioms of Probability

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $AB = 0$
   $\rightarrow P(A+B) = P(A) + P(B)$

How to determine the *p.d.f.* of a random variable (4)





(1) choose a value for Δx (resolution improves as $\Delta x \downarrow$)

(2) choose a value for T (accuracy improves as $T \uparrow$)

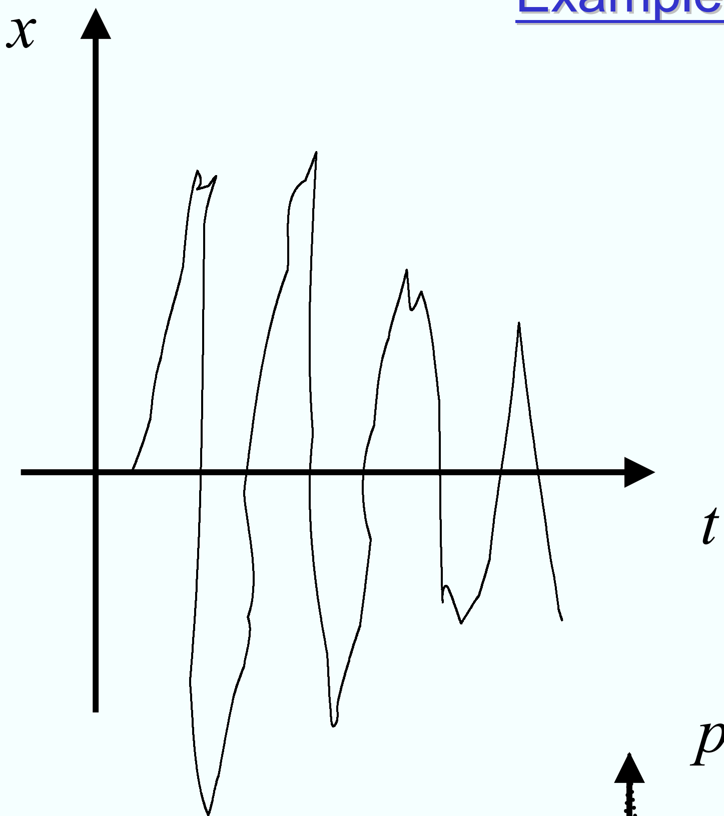
$$\int_{x_1}^{x_2} p(x)\,dx = P(x_1 \le x \le x_2)$$

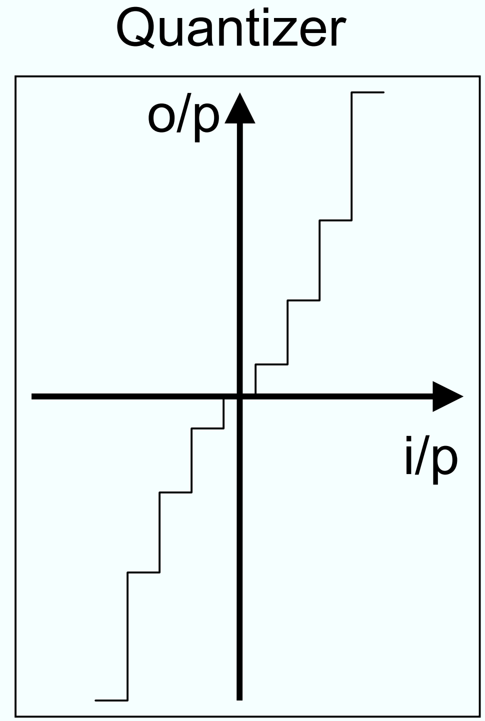
For $x_1 \cong x_2$, $p(x)$ will be approximately constant over that interval

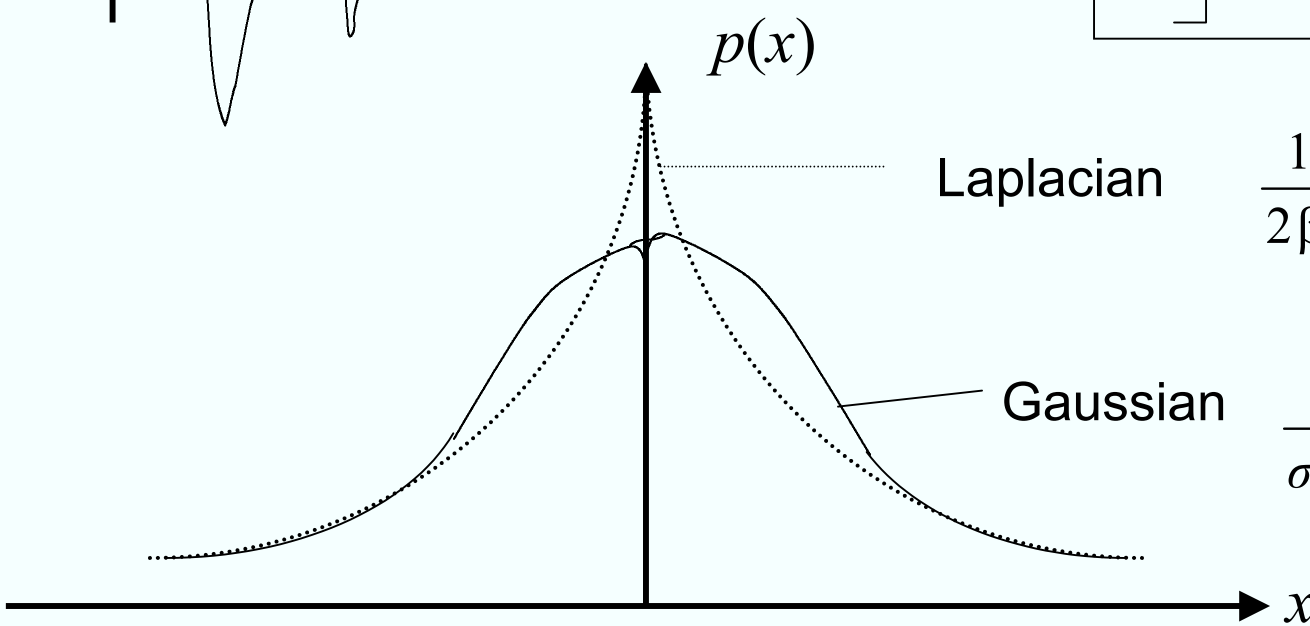
$$p(x)(x_2 - x_1) \cong P(x_1 \le x \le x_2),\ x \in [x_1, x_2]$$

$$p(x) \cong \frac{P(x_1 \le x \le x_2)}{x_2 - x_1},\ x \in [x_1, x_2] \longrightarrow p(x) \cong \frac{1}{\Delta x} \cdot \frac{\sum_{i=1}^{j} \Delta t_i}{T}$$

Examples of *p.d.f.s* (5)







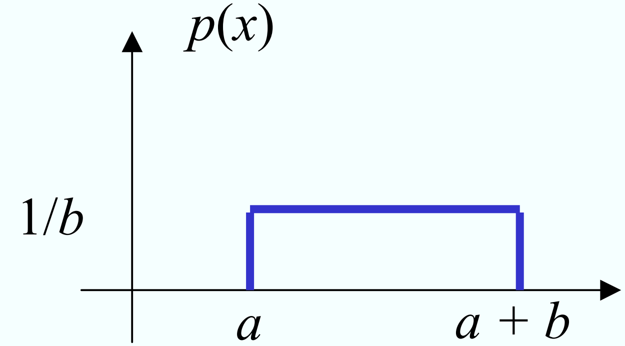
Laplacian: $$\frac{1}{2\beta}\exp\left[\frac{-|x|}{\beta}\right]$$

Gaussian: $$\frac{1}{\sigma\sqrt{2\pi}}\boldsymbol{exp}\left[\frac{-x}{\sigma\sqrt{2}}\right]^2$$

Uniform Distribution

Def: The uniform distribution with parameters a & b is defined by the

$$p.d.f. \quad p(x) = \begin{cases} 1/b & \text{for } x \in [a, a+b] \\ 0 & \text{otherwise} \end{cases}$$



Properties:

$$\mathbf{E}[\boldsymbol{x}] = \overline{\boldsymbol{x}} = \int_{-\infty}^{\infty} x \cdot p(x) dx = \frac{1}{b} \int_a^{a+b} x dx = \frac{1}{b} \left[\frac{x^2}{2} \right]_a^{a+b} = a + \frac{b}{2}$$

$$\mathbf{E}[\boldsymbol{x}^2] = \int_a^{a+b} x^2 \cdot p(x) dx = \frac{1}{b} \left[\frac{x^3}{3} \right]_a^{a+b} = a^2 + ab + \frac{b^2}{3}$$

$$\sigma^2 = \overline{\boldsymbol{x}^2} - \overline{\boldsymbol{x}}^2 = a^2 + ab + \frac{b^2}{3} - a^2 - \frac{b^2}{4} - ab = \frac{b^2}{12}$$

Gaussian Distribution (7)

Def: The Gaussian (normal) distribution with parameters σ & $\bar{x}$ is defined by the *p.d.f.*

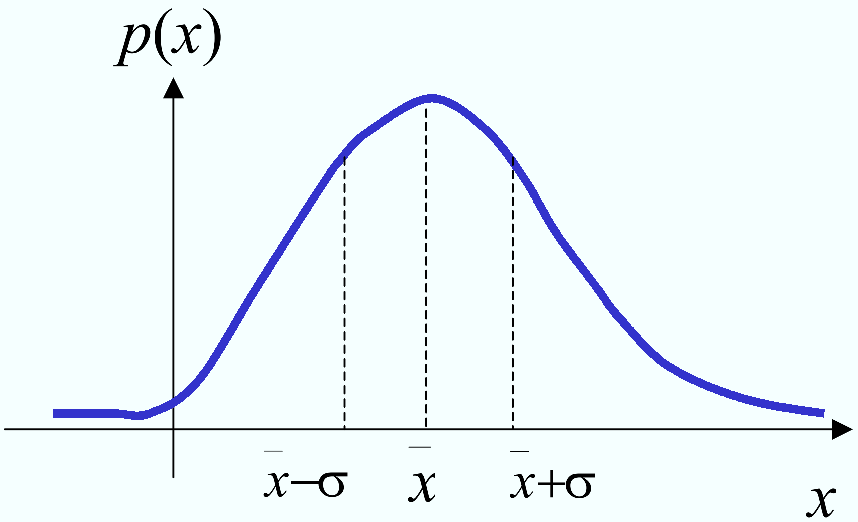
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

Properties:

$$E[\boldsymbol{x}] = \int_{-\infty}^{\infty} x \cdot p(x)dx = \bar{\boldsymbol{x}}$$

$$E[\boldsymbol{x}^2] = \int_{-\infty}^{\infty} x^2 \cdot p(x)dx = \bar{x}^2 + \sigma^2$$

$$\therefore \sigma^2 = E[\boldsymbol{x}^2] - E[\boldsymbol{x}]^2$$



p.d.f. of Linearly Combined Random Variables (8)

$\boldsymbol{y} = a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2 + \ldots\ldots + a_n\boldsymbol{x}_n$ where a_1 , a_2 , …… , a_n are constants,

$\boldsymbol{x}_1$, $\boldsymbol{x}_2$, …….. , $\boldsymbol{x}_n$ are n independent variable &

$p_{x_1}(x_1), p_{x_2}(x_2), \ldots, p_{x_n}(x_n)$ are their $p.d.f$ respectively.

If $\boldsymbol{y} = \boldsymbol{x}_1 + \boldsymbol{x}_2$ then

$$p_y(y) = \int_{-\infty}^{\infty} p_{x_1}(\alpha) \cdot p_{x_2}(y - \alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} p_{x_2}(\alpha) \cdot p_{x_1}(y - \alpha)\, d\alpha$$

$= \text{convolution of } p_{x_1}(x_1) \;\&\; p_{x_2}(x_2)$

$$= p_{x_1} \otimes p_{x_2}$$

$$= p_{x_2} \otimes p_{x_1}$$

If $\boldsymbol{y} = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \ldots + \boldsymbol{x}_n$

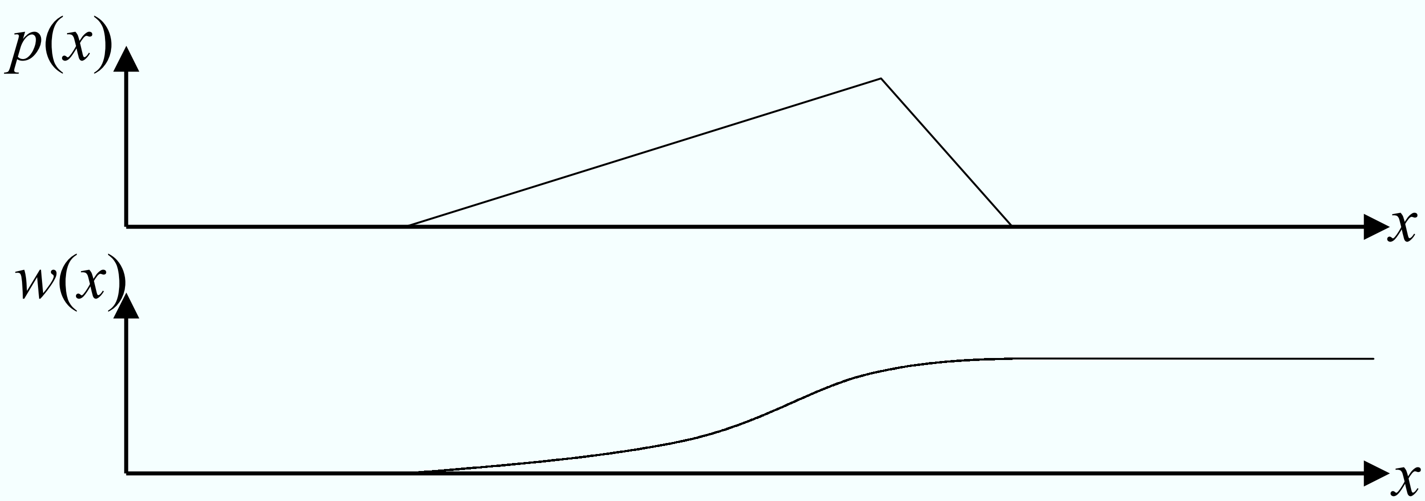
then $p_y(y) = p_{x_1} \otimes p_{x_2} \otimes \cdots \otimes p_{x_n}$

Probability Distribution Function (9)

Def: The Probability Distribution Function $W(x_1)$ of a random variable x is the probability that x is less than or equal to x_1 ,

i.e. $$W(x_1) = \int_{-\infty}^{x_1} p(x)dx = P(x \le x_1).$$

e.g. 1



Random Variables and Distribution Function (10)

Consider a random variable $\boldsymbol{x}$ of $p.d.f.$ $p(x)$ and distribution function $W(x)$.

Note that:

- The x in $p(x)$ and $W(x)$ is not a random variable but a value of the random variable $\boldsymbol{x}$.

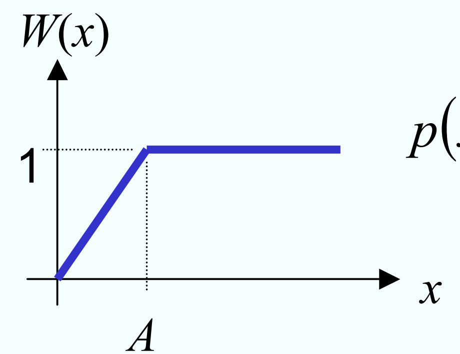
- The $p.d.f.$ $p(x)$ is not a probability but a rate of change of the probability $W(x)$ w.r.t. $\boldsymbol{x}$, i.e. $\dfrac{dW(x)}{dx}$.

- The distribution function $W(x)$ of random variable $\boldsymbol{x}$ is the probability that $\boldsymbol{x}$ has a value less than or equal to the value x.

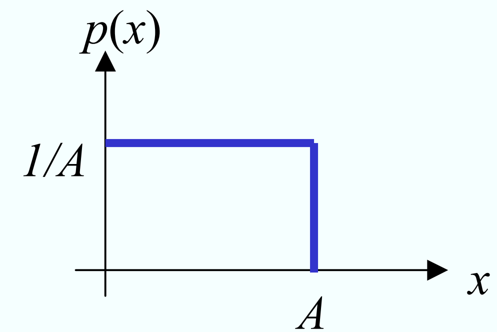
Probability Distribution Function $W(x)$ (11)

$$p(x) = \frac{dW(x)}{dx} \quad \text{or} \quad p(x_1) = \lim_{\Delta x \to 0} \frac{W(x_1 + \Delta x) - W(x_1)}{\Delta x}$$

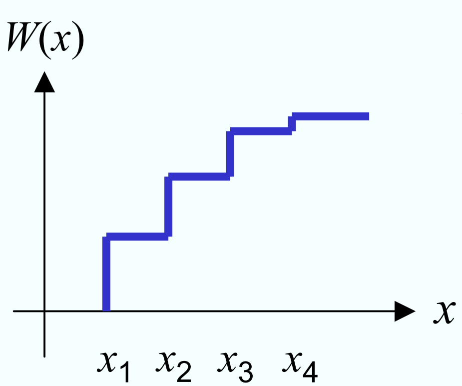
Example 1



$$p(x) = \frac{dW(x)}{dx} = \begin{cases} \frac{1}{A} & x \in (0, A] \\ 0 & \text{otherwise} \end{cases}$$

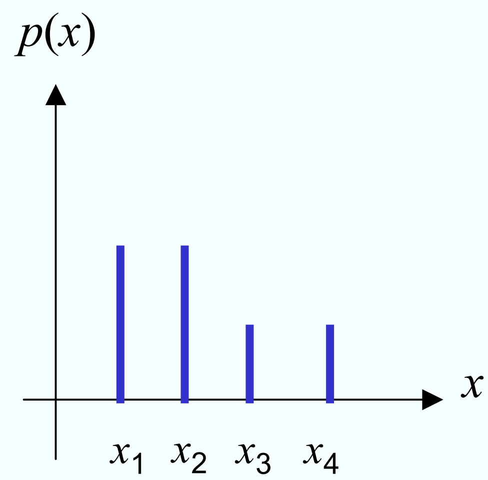


Example 2



$$p(x) = \sum_{i=1}^{4} P(x_i) \cdot \delta(x - x_i)$$

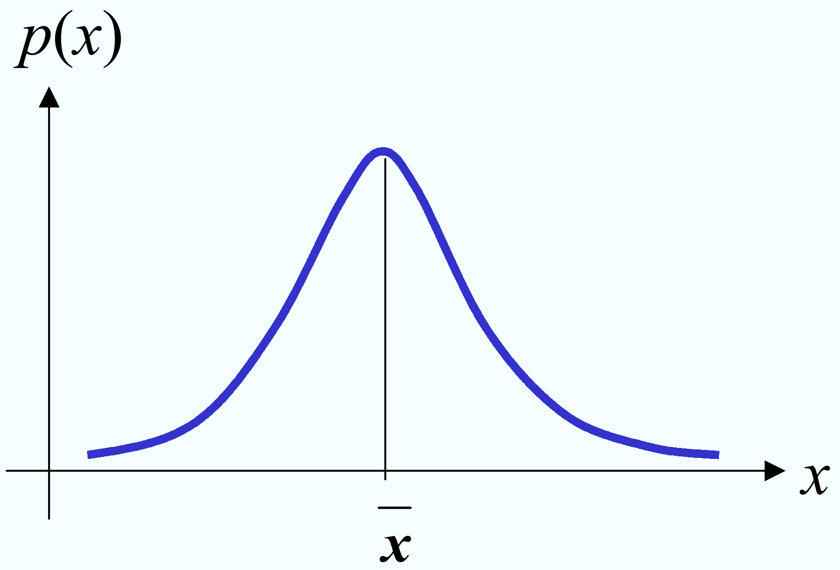
where $P(x_i) = W(x_i) - W(x_i - \varepsilon)$



Statistical Average (12)

Def. The average value $\bar{x}$ of a random variable x with a *p.d.f.* $p(x)$ is

$$\bar{x} = \int_{-\infty}^{\infty} x \cdot p(x)\, dx.$$



Note :

1. $\bar{x} \equiv \mathrm{E}[x] \equiv$ Expected value ≡ Mean value ≡ Ensemble average

2. If $g(x)$ is an arbitrary function of x, the expected value of $g(x)$ is $\mathrm{E}[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot p(x) dx.$

Moments (13)

Def. The n th moment of $p(x)$ (about the origin)

$$\equiv \quad \mathrm{E}\left[\boldsymbol{x}^n\right] = \int_{-\infty}^{\infty} x^n \cdot p(x)\,dx \qquad n = 1, 2, \ldots$$

Def. The n th moment of $p(x)$ about the x_0

$$\equiv \quad \mathrm{E}\left[(\boldsymbol{x} - x_0)^n\right] = \int_{-\infty}^{\infty} (x - x_0)^n \cdot p(x)\,dx \qquad n = 1, 2, \ldots$$

Note: 1. The first moment (n = 1) (about the origin)

$$= \mathrm{E}[\boldsymbol{x}] = \int_{-\infty}^{\infty} x \cdot p(x)\,dx = \overline{\boldsymbol{x}}$$ = mean value of $\boldsymbol{x}$

2. The second moment (n = 2)

$$= \mathrm{E}\left[\boldsymbol{x}^2\right] = \int_{-\infty}^{\infty} x^2 \cdot p(x)\,dx = \overline{\boldsymbol{x}^2}$$ = mean square value of $\boldsymbol{x}$

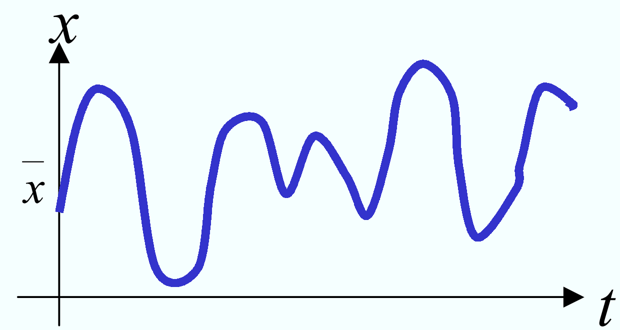
Given : a r.v. x & its mean is $\bar{x}$

i) *Find:* $E\left[\left(x-\bar{x}\right)\right]$

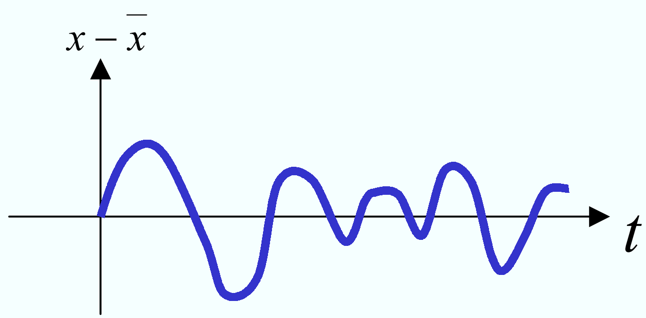
Sln: $E\left[\left(x-\bar{x}\right)\right] = \int_{-\infty}^{\infty}\left(x-\bar{x}\right)p(x)dx$

$$= \int_{-\infty}^{\infty} x\cdot p(x)dx - \bar{x}\int_{-\infty}^{\infty} p(x)dx$$

$$= 0$$



ii) *Find:* $E\left[\left(x-\bar{x}\right)^2\right]$



Sln: $E\left[\left(x-\bar{x}\right)^2\right]$

$$= E\left[x^2 + \bar{x}^2 - 2\,x\,\bar{x}\right]$$

$$= \int_{-\infty}^{\infty} x^2\cdot p(x)dx + \bar{x}^2\int_{-\infty}^{\infty} p(x)dx - 2\bar{x}\int_{-\infty}^{\infty} x\cdot p(x)dx$$

$$= \overline{x^2} + \bar{x}^2 - 2\bar{x}^2$$

$\therefore\ \overline{\left(x-\bar{x}\right)^2} = \overline{x^2} - \bar{x}^2 \equiv \sigma^2 =$ variance of the r.v. x

Power of the r.v. (A.C. +D.C.)

Power of the D.C. component

Power of the A.C. component

Central Limit Theorem (15)

$\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are independent random variables with $p.d.f.$ $p_{x1}, p_{x2}, \ldots, p_{xn}$. For $\boldsymbol{y} = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \ldots + \boldsymbol{x}_n$, the $p.d.f.$ of $\boldsymbol{y}$ is

$$p_y(y) = p_{x1} \otimes p_{x2} \otimes \cdots \otimes p_{xn}$$ where $\otimes$ is convolution.

Th^m: If n is very large, then for all p_{xi} the $p.d.f.$ of $\boldsymbol{y}$ equals

$$\lim_{n\to\infty} p_y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-\bar{y})^2}{2\sigma^2}}$$

where

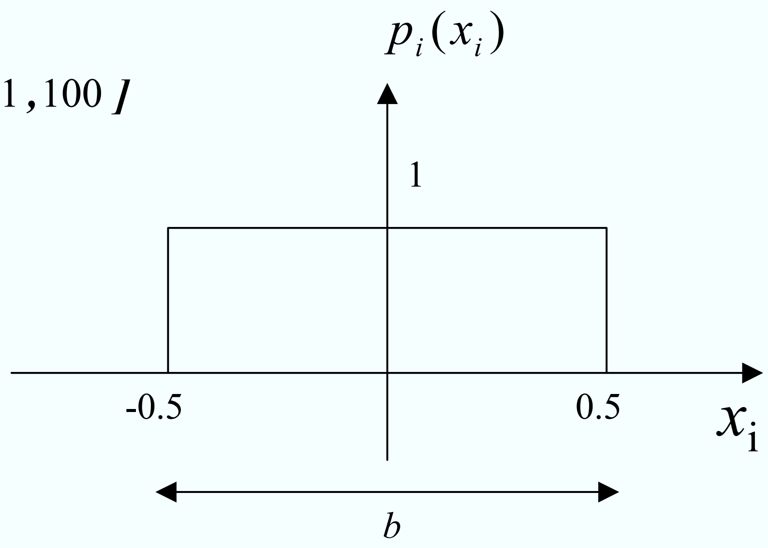
$$\overline{\boldsymbol{y}} = \overline{\boldsymbol{x}_1} + \overline{\boldsymbol{x}_2} + \ldots + \overline{\boldsymbol{x}_n}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$$

Central Limit Theorem (16)

Given: $\boldsymbol{y} = \sum_{i=1}^{100} \boldsymbol{x}_i$

$$p_i(x_i) = \begin{cases} 1 & x \in [-0.5, 0.5] \\ 0 & otherwise \end{cases} \qquad for\ i \in [\,1, 100\,]$$



Find: $p_y(y)$ the *p.d.f.* of $\boldsymbol{y}$

Sln:

$$\overline{x_i} = 0$$

$$\sigma_i^2 = \frac{b}{12} = \frac{1}{12}$$

$$\overline{\boldsymbol{y}} = \sum_{i=1}^{100} \overline{\boldsymbol{x}_i} = 0$$

$$\sigma^2 = \sum_{i=1}^{100} \sigma_i^2 = \frac{100}{12}\ or\ \frac{25}{3}$$

$$p_y(y) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{\frac{-(y-\overline{y})^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{\frac{50\pi}{3}}} \cdot e^{\frac{-3y^2}{50}}$$

Def. The joint probability density function of two random variables $\boldsymbol{x}$ and $\boldsymbol{y}$ is a function $p(x,y)$ that possesses the properties

$$i)\quad p(x,y) \geq 0$$

$$ii)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 1$$

$$iii)\quad P(x_1 \leq x \leq x_2, y_1 \leq y \leq y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} p(x,y)\,dx\,dy$$

Def. The joint probability distribution function is

$$W(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} p(x,y)\,dx\,dy$$

so $$p(x,y) = \frac{\partial^2 W(x,y)}{\partial x\,\partial y}.$$

Def. The random variables $\boldsymbol{x}$ and $\boldsymbol{y}$ with *p.d.f.* $p_x(x)$ and $p_y(y)$ are independent if $p(x,y) = p_x(x)\,p_y(y)$.

Two–Dimensinal Distributions (18)

Def. The maginal probabiliy density functions of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are $p_1(x) = \int_{-\infty}^{\infty} p(x,y)\,dy$ & $p_2(y) = \int_{-\infty}^{\infty} p(x,y)\,dx.$

Def. The maginal probability distribution functions are

$$W_1(x) = \int_{-\infty}^{x} p_1(x)\,dx = \int_{-\infty}^{x}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx$$

$$W_2(y) = \int_{-\infty}^{y} p_2(y)\,dy = \int_{-\infty}^{y}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy$$

Find: k for the 2D *p.d.f.*

$$p(x,y) = \begin{cases} k\,e^{-2x-3y} & x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Sln:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 1$$

$$\rightarrow k\int_{0}^{\infty} e^{-2x}dx \int_{0}^{\infty} e^{-3y}dy = 1$$

$$\rightarrow k \cdot \frac{1}{2} \cdot \frac{1}{3} = 1$$

$$\therefore k = 6$$

Moments of 2-D *p.d.f.* (19)

Def. The moments of a joint *p.d.f.* $p(x,y)$ are called joint moments

$$\mu'_{i,j} = \mathrm{E}[\boldsymbol{x}^i \, \boldsymbol{y}^j] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i y^j \, p(x,y)\, dx\, dy$$

where $i, j = 0, 1, 2, 3, \ldots$ and the order of μ'_{ij} is $i+j$.

Note:

$$\mathrm{E}[\boldsymbol{x}] = \overline{\boldsymbol{x}} = \mu'_{1,0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i \, p(x,y)\, dx\; dy$$

$$\mathrm{E}[\boldsymbol{y}] = \overline{\boldsymbol{y}} = \mu'_{0,1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^j \, p(x,y)\, dx\; dy$$

Def. The central moments (i.e. moments about the mean) are

$$\mu_{ij} = \mathrm{E}[(\boldsymbol{x}-\overline{\boldsymbol{x}})^i(\boldsymbol{y}-\overline{\boldsymbol{y}})^j] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\overline{x})^i\,(y-\overline{y})^j\, p(x,y)\, dx\; dy$$

where $i, j = 0, 1, 2, 3, \ldots$ and the order of μ'_{ij} is $i+j$.

Note: The moment μ_{11} is called the covariance of two variables.

c.f. variance of $\boldsymbol{y}$ is $\sigma_y^2 = \int_{-\infty}^{\infty}(y-\overline{y})^2\, p_y(y)\, dy$

Example

Find: the three 2^{nd} order moments of the r.v.s $\boldsymbol{x}$ and $\boldsymbol{y}$.
Sln:

The three 2^{nd} order moments are μ_{20}, μ_{02} and μ_{11}.

$$\mu_{20} = \mathrm{E}[(\boldsymbol{x}-\overline{\boldsymbol{x}})^2] = \mathrm{E}[\boldsymbol{x}^2 - 2\boldsymbol{x}\overline{\boldsymbol{x}} + \overline{\boldsymbol{x}}^2] = \overline{\boldsymbol{x}^2} - \overline{\boldsymbol{x}}^2$$

$$\mu_{02} = \mathrm{E}[(\boldsymbol{y}-\overline{\boldsymbol{y}})^2] = \mathrm{E}[\boldsymbol{y}^2 - 2\boldsymbol{y}\overline{\boldsymbol{y}} + \overline{\boldsymbol{y}}^2] = \overline{\boldsymbol{y}^2} - \overline{\boldsymbol{y}}^2$$

$$\therefore \mu_{20} = \sigma_x^2 \qquad\qquad \mu_{02} = \sigma_y^2$$

$\mu_{11} =$ covariance

$$= \mathrm{E}[(\boldsymbol{x}-\overline{\boldsymbol{x}})(\boldsymbol{y}-\overline{\boldsymbol{y}})]$$

$$= \mathrm{E}[\boldsymbol{xy} - \boldsymbol{x}\,\overline{\boldsymbol{y}} - \overline{\boldsymbol{x}}\,\boldsymbol{y} + \overline{\boldsymbol{x}}\,\overline{\boldsymbol{y}}]$$

$$= \overline{\boldsymbol{xy}} - \overline{\boldsymbol{x}}\,\overline{\boldsymbol{y}}$$

Correlation

Def. The numerical measure of the *similarity* between x and y is the normalised correlation coefficients and is defined as

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{E[(\boldsymbol{x}-\bar{\boldsymbol{x}})(\boldsymbol{y}-\bar{\boldsymbol{y}})]}{\sqrt{E[(\boldsymbol{x}-\bar{\boldsymbol{x}})^2\; E[(\boldsymbol{y}-\bar{\boldsymbol{y}})^2]}}.$$

Note: (i) $\rho \in [-1, 1]$

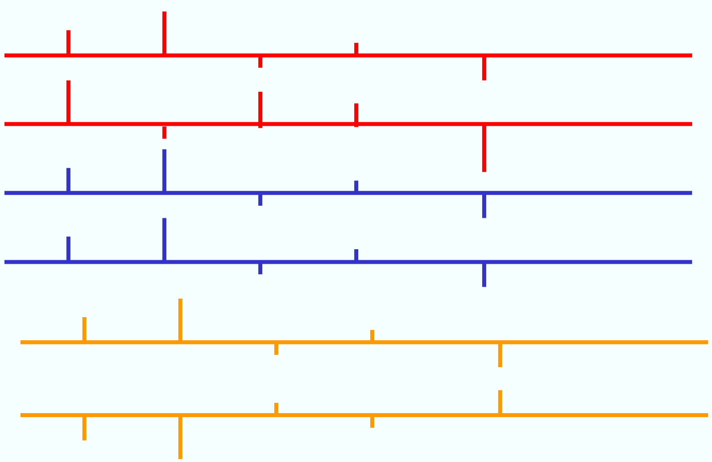
(ii) $\rho = 0$ if $\boldsymbol{x}$ and $\boldsymbol{y}$ are uncorrelated (i.e. $\mu_{11} = 0$).

e.g.

$\rho \approx 0 \begin{cases} x \\ y \end{cases}$

$\rho \rightarrow 1 \begin{cases} x \\ y \end{cases}$

$\rho \rightarrow -1 \begin{cases} x \\ y \end{cases}$



Def: Random variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are uncorrelated if $\rho = 0$ or $\mu_{11} = 0$.

Independent and uncorrelated

Def: Random variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are uncorrelated if $\rho = 0$ or $\mu_{11} = 0$.

Th^m: If $\boldsymbol{x}$ and $\boldsymbol{y}$ are statistically independent then they are uncorrelated.

Proof: $\boldsymbol{x}$ and $\boldsymbol{y}$ are statistically independent so $p(x,y) = p_1(x)\, p_2(y)$ where $p_1(x)$ and $p_2(y)$ are *p.d.f.* of $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively.

$$\rightarrow \overline{\boldsymbol{xy}} = \mathrm{E}[\boldsymbol{xy}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\; p(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x p_1(x)\, dx \int_{-\infty}^{\infty} y p_2(y)\, dy$$

$$= \overline{\boldsymbol{x}}\;\overline{\boldsymbol{y}} \qquad \therefore\ \mu_{11} = 0$$

note:

$$\mu_{11} = \mathrm{E}[(x-\overline{x})(y-\overline{y})]$$

$$= \overline{xy} - \overline{x}\,\overline{y}$$

$$\rightarrow \mathrm{E}[(x-\overline{x})(y-\overline{y})] = \mathrm{E}[xy - x\overline{y} - \overline{x}y + \overline{x}\,\overline{y}] = \overline{xy} - \overline{x}\,\overline{y} = 0$$

$\rightarrow$ $\mu_{11} = 0$ and so $\boldsymbol{x}$ and $\boldsymbol{y}$ are uncorrelated.

Examples

Find: μ_{22} if $p(x,y) = p_1(x)\ p_2(y)$

Sln:

$$\mu_{22} = \mathrm{E}[(x-\bar{x})^2(y-\bar{y})^2]$$

$$= \mathrm{E}[(x-\bar{x})^2]\ \mathrm{E}[(y-\bar{y})^2]$$

$$= (\overline{x^2} - \bar{x}^2)\,(\overline{y^2} - \bar{y}^2)$$

$$= \overline{x^2}\,\overline{y^2} - \overline{x^2}\,\bar{y}^2 - \bar{x}^2\,\overline{y^2} + \bar{x}^2\,\bar{y}^2$$

Given: A r.v. $\boldsymbol{x}$ is uniformly distribution between -1 and +1.

Find: the normalised correlation coefficient for x and y if $y = x^2$.

Sln:

$$\mu_{11} = \overline{x\,y} - \bar{x}\,\bar{y}$$

$$= \overline{x^3} - \bar{x}\,\overline{x^2}$$

remind: $\rho = \dfrac{\mu_{11}}{\sqrt{\mu_{20}\ \mu_{02}}}$

$$\bar{x} = \int_{-\infty}^{\infty} x\ p(x)\,dx = 0$$

$$\overline{x^3} = \int_{-\infty}^{\infty} x^3\ p(x)\,dx = 0 \qquad \therefore\ \ \mu_{11} = 0 \quad \rightarrow \quad \rho = 0$$

Example

Given: ***x*** and ***y*** are *2* independent r.v.s and $u = x + y$ and $v = x - y$

Find: the condition under which ***u*** and ***v*** are uncorrelated.

Sln:

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 \qquad \sigma_y^2 = \overline{y^2} - \bar{y}^2$$

$$\mu_{11} = \mathrm{E}[(u - \bar{u})(v - \bar{v})]$$

$$= \mathrm{E}[\{(x+y) - \overline{(x+y)}\}\{(x-y) - \overline{(x-y)}\}]$$

$$= \mathrm{E}[x^2 - y^2 + \bar{x}^2 - \bar{y}^2 - (x-y)(\bar{x}+\bar{y}) - (x+y)(\bar{x}-\bar{y})]$$

$$= \overline{x^2} - \overline{y^2} + \bar{x}^2 - \bar{y}^2 - (\bar{x}-\bar{y})(\bar{x}+\bar{y}) - (\bar{x}+\bar{y})(\bar{x}-\bar{y})$$

$$= (\overline{x^2} - \overline{y^2}) - (\bar{x}^2 - \bar{y}^2)$$

$$= \sigma_x^2 - \sigma_y^2 \quad \therefore\ \mu_{11} = 0 \ \ \mathit{if} \ \ \sigma_x = \sigma_y$$