Foundametnal Course on

Probability, Random Variable and Random Processes

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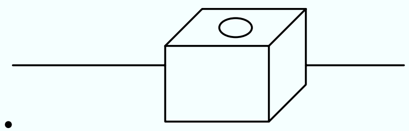
- Probability Theory
- Random Variables
- Random Processes
  - Stationary RP & Ergodic RP
  - Gaussian RP
  - Filtering of RP

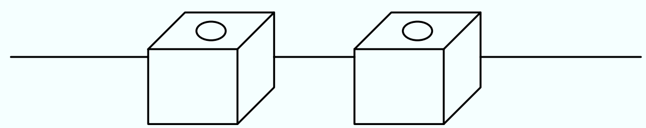
Probability Theory - 3 stages

In order to develop a useful theory of probability, it is important to separate 3 stages in the consideration of any real probability.

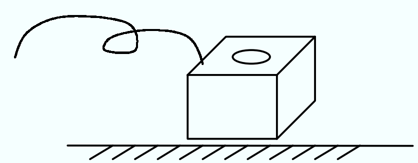
1. The association of an events with a probability by (i) experiments and (ii) reasoning.

e.g.  $P(1) = 1/6$

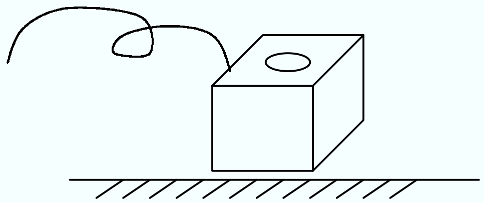
2. Development of the relationship of the probability of an event with the probabilities of some other events.

e.g.  $P(1)\,P(1)$

3. The application of the results of stage 1 & stage 2 to the real world.

e.g. 

Probability Theory - some definitions

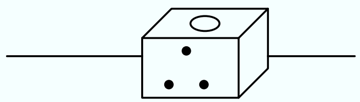


discrete *
continuous

A single performance is called a trial & at it we observe a single outcome S_i.

The event A is said to have occurred in this trial if $S_i \in A$.

e.g.



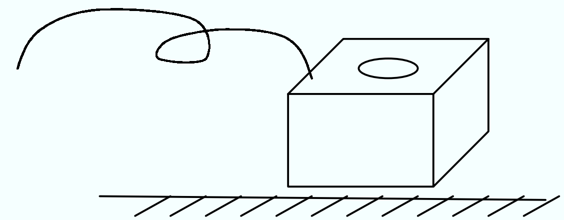
outcome $S_i = 1$
event $A = \{1,2,3\}$ occurred.

Probability space S = universal set = Set contains all possible experimental outcomes

e.g. $S = \{1,2,\ldots,6\}$

ϕ = empty set = set contains impossible outcomes

Stage 1: The association of an events with a probability



For example, we are to determine the probability of event A which is the outcome being *one* in a trial of throwing a dice.

- Probability determined by experiment

The experiment of throwing a dice is repeated n times. Suppose n_a times of the n trials result in event A.

Probability of A = $P(\mathrm{A}) \equiv \dfrac{n_a}{n}$ provided n is sufficiently large.

Comments:

(i) not exact!

(ii) $\lim\limits_{n \to \infty} \dfrac{n_a}{n} \equiv P(\mathrm{A})$ may be exact but cannot be found in practice.

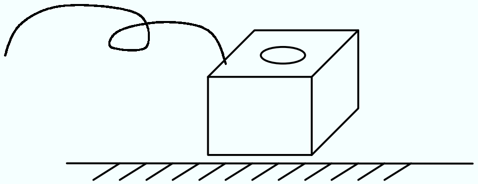
- Probability determined by reasoning

We may assume that throwing a dice has six possible outcomes and so there are 6 possible events. If all events have the same probability, then $P(\mathrm{A})$ = $1/6$.

Comment: not exact as the assumptions may be wrong.

Stage 2: Development of the relationship between probabilities of different events

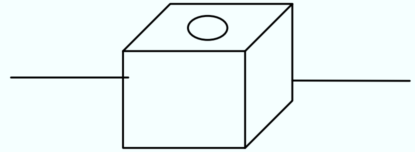
a trial



Space S

$\{1,2,\ldots,6\}$

outcome = 1



Event {1} occurred.

We assign to each event A a number $P(A)$ which we call the probability of A. This number satisfies the 3 axioms:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $AB = \phi \rightarrow P(A+B) = P(A) + P(B)$ (i.e. mutually exclusive)

Th$\underline{^m}$.

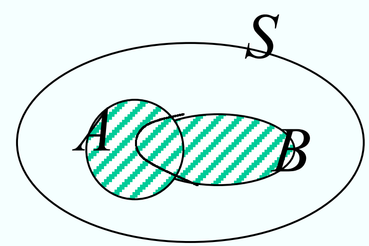
1. $P(\overline{A}) = 1 - P(A)$
2. $P(\phi) = 0$
3. $AB \neq 0 \rightarrow P(A+B) = P(A) + P(B) - P(AB)$

   $\therefore P(AB) \leq P(A) + P(B)$
4. $B \subset A \rightarrow P(A) = P(B) + P(A\overline{B}) \geq P(B)$

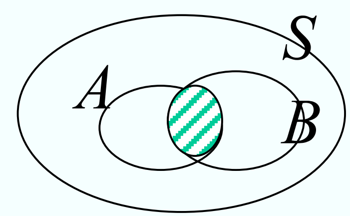
Fill in the missing words in ____

Definitions:-

1) Event S (universal set) occurs at ____ trial.
2) Event ϕ (empty set) ____ occurs.
3) Event $A+B$ occurs when event A ____ event B occur.

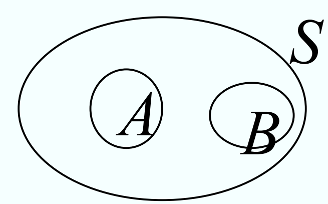


4) Event AB occurs when event A ____ event B occur.



Properties:-

1) $A \cdot B = 0 \longrightarrow$ event A & event B ____ occur in the same trial.



2) Event A occurs $\longrightarrow$ Event $\overline{A}$ ____ occurs.

Two Theorems in Probability Theory

Prove: $P(\overline{A}) = 1 - P(A) \leq 1$

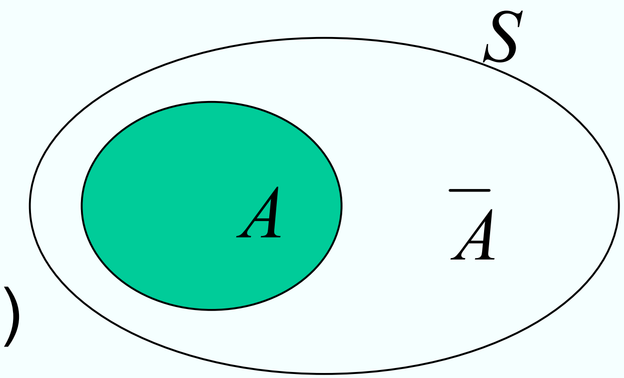
Proof:

$A \cdot \overline{A} = \phi$ (set theory)

$A + \overline{A} = S$ (set theory)

$\therefore P(A + \overline{A}) = P(A) + P(\overline{A}) = 1$ (axiom 3)

$\rightarrow P(\overline{A}) = 1 - P(A)$ Q.E.D.



Prove: $P(\phi) = 0$

Proof:

$S = S + \overline{S} = S + \phi$ (set theory)

$S \cdot \overline{S} = \phi$ (set theory)

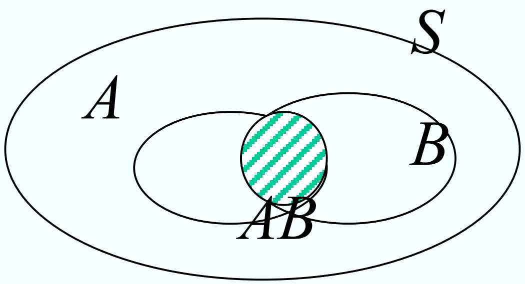
$\therefore P(S + \phi) = P(S) + P(\phi) = 1$

$\rightarrow P(\phi) = 1 - P(S) = 0$ Q.E.D.

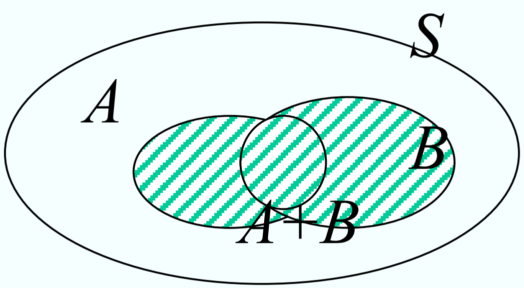
Three definitions in Probability Theory

Given event A & event B, we have

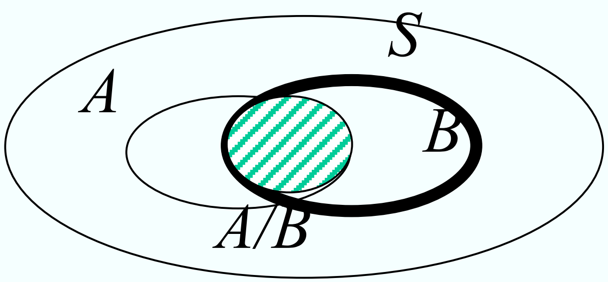
$P(AB)$ The probability of the occurrence of events A & B.



$P(A+B)$ The probability of the occurrence of event A or B.



$P(A/B)$ The probability of the occurrence of event A given B.



In general, we have

$$P(ABCD....) \quad \&$$

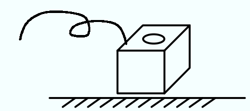
$$P(A+B+C+D+....)$$

Probability Theory

Def. The conditional event for A given B, A/B, is the event A under the stipulation that B has occurred.

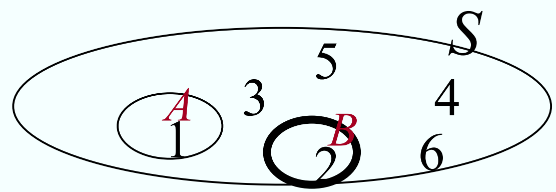
Def. The conditional probability of A given B is $P(A/B) \equiv P(AB)/P(B)$.

Examples



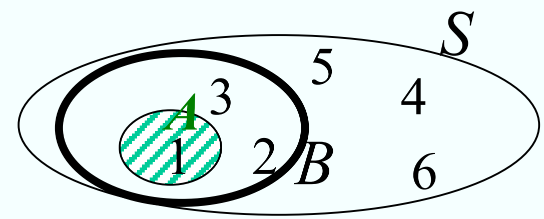
Properties

1) $AB = 0 \rightarrow P(A/B) = 0$



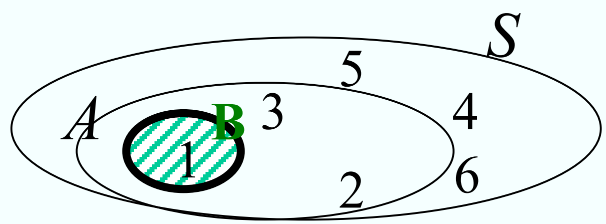
2) $A \subset B \rightarrow A \cdot B = A$

$$\rightarrow P(A/B) = \frac{P(A)}{P(B)} \geq P(A)$$



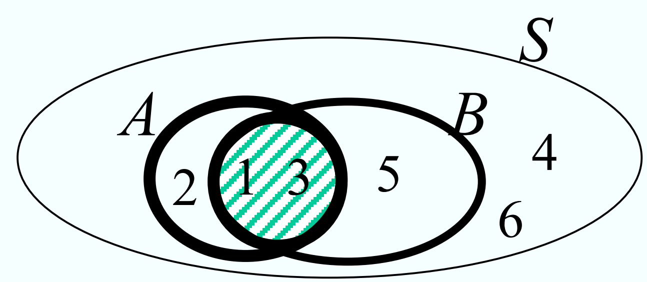
3) $B \subset A \rightarrow AB = B$

$$\rightarrow P(A/B) = \frac{P(B)}{P(B)} = 1$$



4) $A = \{1,2,3\}$
$B = \{1,3,5\}$

$$\rightarrow P(A/B) = \frac{P(AB)}{P(B)} = \frac{1/3}{1/2} = \frac{2}{3}$$



Probability Theory

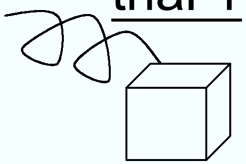
Def. Two events A & B are independent if $P(AB) \equiv P(A)P(B)$.

Two events are independent when knowledge of the occurrence of one event gives no additional information concerning the likelihood of the occurrence of a second event.

$$P(A/B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

e.g.

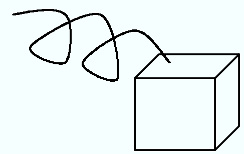
<u>trial 1</u>



$S_1 = \{1,2,3,4,5,6\}$

$A_1 = \{1\}$ $\quad P(A_1) = \frac{1}{6}$

<u>trial 2</u>



$S_1 = \{1,2,3,4,5,6\}$

$A_2 = \{1\}$ $\quad P(A_2) = \frac{1}{6}$

$P(\underline{A_1}\underline{A_2}) = P(A_1)P(A_2)$ if A_1 & A_2 are independent.

The space of A_1A_2 is
$S = S_1 \times S_2$
$= \{ (1,1), (1,2), (1,3), \ldots, (6,6)\}$.

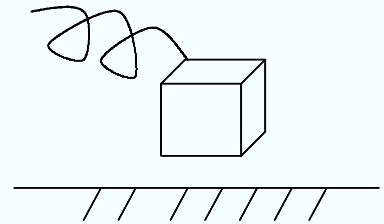
Event A_1A_2 consisting of all ordered-pairs (S_{1i}, S_{2j}) s.t. $S_{1i} \in A_1$ $S_{2j} \in A_2$ is a subset of S.

Mutually Exclusive Events and Independent Events

e.g. 1 $A = \{1\}$ $B = \{2\}$

$$P(A/B) = \frac{P(AB)}{P(B)} = \frac{P(\phi)}{P(B)} = 0$$

a single trial

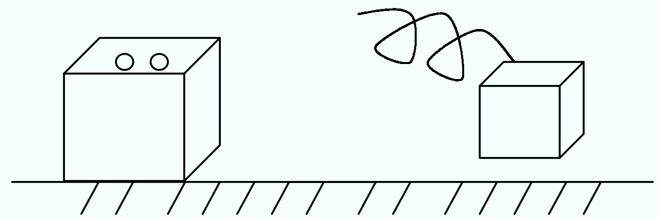


A & B are mutually exclusive,
i.e. $AB=0$

e.g. 2 $A = \{1\}$ $B = \{2\}$

$$P(A/B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

trial 1 trial 2



A & B are independent,
i.e. $P(AB)=P(A)P(B)$

Independent Events

Def. Events A_1, A_2 & A_3 are independent

iff $P(A_1A_2A_3) = P(A_1)\,P(A_2)\,P(A_3)$

& $P(A_1A_2) = P(A_1)\,P(A_2)$

& $P(A_1A_3) = P(A_1)\,P(A_3)$

& $P(A_2A_3) = P(A_2)\,P(A_3)$.

Def. In general, n events $A_1, A_2, \ldots, A_n$ are independent iff

$P(A_1A_2 \ldots A_n) = P(A_1)\,P(A_2) \ldots P(A_n)$

$\vdots \qquad\qquad \vdots$

$P(A_iA_jA_k) = P(A_i)\,P(A_j)\,P(A_k)$

$P(A_iA_j) = P(A_i)\,P(A_j)$

for all combinations of $i, j, k, \ldots$

where $1 \leq i \leq j \leq k \leq \ldots \leq n$.

Example:

Given:

$P(A_1) = 1/2$

$P(A_2) = 1/4$

$P(A_3) = 1/4$

$P(A_1A_2) = 1/8$

$P(A_1A_3) = 1/8$

$P(A_2A_3) = 1/8$

$P(A_1A_2A_3) = 1/32$

Are A_1, A_2 & A_3 independent?

*Th^m^*. For events $A_1, A_2, \ldots, A_i$ (which may or may not be independent), the probability of the simultaneous occurrence of the i events is

$$P(A_1A_2\ldots A_i) = P(A_1)\cdot P(A_2/A_1)\cdot P(A_3/A_1A_2)\cdots P(A_i/A_1A_2\ldots A_{i-1}).$$

Example: Let A_1, A_2, A_3 & A_4 represent the consecutive events of drawing an aces.

Find: $P(A_1A_2A_3A_4)$

Solution:

$$P(A_1A_2A_3A_4) = P(A_1)P(A_2/A_1)P(A_3/A_1A_2)P(A_4/A_1A_2A_3)$$

$$= \frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}\cdot\frac{1}{49}$$

$$= \frac{1}{270725}$$

PROBABILITY THEORY

set theory

AB

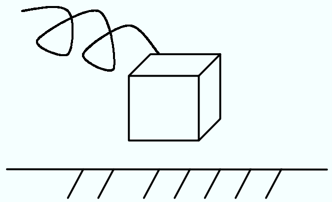
$A + B$

S

ϕ

Defintions:

trial outcome event



1 $\{1\}$

3 *axioms*

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $AB = 0 \rightarrow$

$$P(A+B) = P(A) + P(B)$$

single trial

Definition: Conditional Probability

$$P(A/B) = \frac{P(AB)}{P(B)}$$

Theorem of Total Probability

$$P(B) = P(B/A_1)P(A_1) + \ldots + P(B/A_n)\cdot P(A_n)$$

Multiplicative Law

$$P(A_1 A_2 \ldots A_n) = P(A_1)\cdot P(A_2/A_1)$$

multiple trials

Definition: Events A & B are independent

$$\underbrace{P(AB)}_{\text{new space}} = \underbrace{P(A)}_{\text{space } S_1} \underbrace{P(B)}_{\text{space } S_2}$$

new space $S = S_1 X S_2$