SURE-LET Image Deconvolution using Multiple Wiener Filters

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Summary

We propose a novel deconvolution algorithm based on the minimization of Stein’s unbiased risk estimate (SURE). We linearly parametrize the deconvolution process by using multiple Wiener filterings as elementary functions, followed by undecimated Haar-wavelet thresholding. The key contributions of our approach are: 1) the linear combination of several Wiener filters with different (but fixed) regularization parameters, which avoids the manual adjustment of a single nonlinear parameter; 2) the use of linear parameterization, which makes the SURE minimization finally boil down to solving a linear system of equations, leading to a very fast and exact optimization of the whole deconvolution process.

Problem statement

Linear observation model:

\[ y = Hx + n \]

where \( H \) is the convolution matrix, Gaussian noise \( n \sim \mathcal{N}(0, \sigma^2) \).

Problem: How to estimate \( x \) from the observations \( y \), knowing \( H \)?

![Image of deconvolution process](image)

SURE for deconvolution problems

Formulation — minimization of MSE

Denoting the processing of the measure data \( y \) by \( F \), our objective is to minimize the mean squared error (MSE): \[ \text{MSE} = \frac{1}{N} \left\{ \left\| F(y) - y \right\|_2^2 \right\} \]

the estimated data — the outcome of the processing \( F \)

SURE — unbiased estimate of MSE

Theorem Given the linear model above, the following random variable:

\[ \epsilon = \frac{1}{N} \left\{ \left\| F(y) \right\|_2^2 - 2y^H F(y) + 2\sigma^2 \text{div}_y \left( H^H F(y) \right) \right\} + \frac{1}{2} \left\| x \right\|_2^2 \]

neutral w.r.t. optimization is an unbiased estimator of the MSE, i.e., \( \mathbb{E}[\epsilon] = \frac{1}{N} \left\{ \left\| F(y) \right\|_2^2 - \left\| y \right\|_2^2 \right\} \), where the divergence operator is \( \text{div}_y = \sum_{k=1}^N \frac{\partial y^k}{\partial u_k} \) for \( \forall u \in \mathbb{R}^N \).

The SURE-LET approach

Regularized SURE — an approximation of SURE

Considering the possible ill-posedness of the matrix \( H \), we approximate \( H^H \) by a Tikhonov regularized inverse \( H^H_{\lambda} \):

\[ e_\lambda = \frac{1}{N} \left\{ \left\| F(y) \right\|_2^2 - 2y^H H^H_{\lambda} F(y) + 2\sigma^2 \text{div}_y \left( H^H_{\lambda} F(y) \right) \right\} + \frac{1}{2} \left\| x \right\|_2^2 \]

where \( H^H_{\lambda} = (H^H + \beta S^H S)^{-1} H^H \) for some \( \beta \) and matrix \( S \), to stabilize \( \epsilon \). In this work, we choose \( \beta = 1 \times 10^{-3} \) and \( S = \text{Laplacian} \) operator.

Linear parameterization of the processing \( F \) — LET

The processing \( F(y) \) is represented by a linear combination of a small number (\( K \ll N \)) of known basic processing \( F_k(y) \) \( \in \mathbb{R}^N \), weighted by unknown linear coefficients \( a_k \) for \( k = 1, 2, \ldots, K \).

\[ F(y) = \sum_{k=1}^K a_k F_k(y) \]

The SURE-LET optimization

Combining SURE and LET, the minimization of \( e_\lambda \) over the unknown linear weights \( a_k \) boils down to solving a linear system of equations of order \( K \):

\[ \sum_{k=1}^K \left\{ \begin{array}{l} (F_k(y) F_k(y)) a_k \; - \; \left\| \begin{array}{c} y H_{\lambda}^H F_k(y) \; - \; \sigma^2 \text{div} \left( H_{\lambda} F_k(y) \right) \end{array} \right\| \; = \; 0 \\ M_{\lambda} c \end{array} \right. \]

Advantage of SURE-LET approach:

1. dramatically reduce the deconvolution problem size from pixel number \( N \) to the number of basis functions \( K \);
2. simplify the deconvolution problem to solving a linear system of equations.

Construction of the functions \( F_k(y) \)

Multi-Wiener deconvolutions

Each basic processing \( F_k \) is Wiener filtering with regularization parameter \( \lambda_k \): \[ F_k(y) = \left( \frac{H_k^H F_k(y) + \lambda_k S_k}{H_k^H S_k + \lambda_k} \right) H_k^H y \]

![Graph of multi-Wiener deconvolution](image)

Experimental results

Parameter setting of the proposed SURE-LET algorithm

- \( \lambda_1 = 1 \times 10^{-4} \)
- \( \lambda_2 = 1 \times 10^{-5} \)
- \( \lambda_3 = 1 \times 10^{-4} \)
- \( \lambda_4 = 3 \times 10^{-5} \)
- \( \beta = 4 \)
- \( M = \text{MM} + \text{LET} \)

Image mixture deconvolution performance (PSNR in dB)

![Graph of mixture deconvolution performance](image)

Visual example

![Example image](image)

Conclusion

- The framework of the proposed SURE-LET approach:
  - extension of SURE to deconvolution problem as the objective functional;
  - linear parameterization of the processing;
  - the originality of the presented work:
    - to use multiple Wiener filterings with different but fixed regularization parameters, to avoid empirical adjustment.
  - the potential of the presented work:
    - great flexibility: take advantage of all the degrees of freedom in the design of the elementary function \( F_k \);
    - limited computational cost: fast and exact to solve a linear system of equations;
    - robustness: to all noise levels.

Reference