**Summary**

We consider the problem of estimating a fractional Brownian motion known only from its noisy samples at the integers. We show that the optimal estimator can be expressed using a digital Wiener-like filter followed by a simple time-variant correction accounting for nonstationarity. Moreover, we prove that this estimate lives in a symmetric fractional spline space and give a practical implementation for optimal upsampling of noisy fBm samples by integer factors.

**What is a fractional Brownian motion?**

A non-stationary zero-average Gaussian random process $W_c(t)$ such that $W_c(0) = 0$ almost surely and whose increments are stationary with

$$d(W_c(t) - W_c(t')) = C(t - t')^\gamma$$

for some $\gamma \in [0, 1]$ (Hurst exponent). It can also be defined more explicitly, either by the characteristic function of any of its measurements $(W_c, \psi)$ (Gelfand-Vilenkin’s distributional approach)

$$\phi(t) = \exp \left( \frac{t^2}{2} \int \frac{\psi^2(\omega) - \psi(0)^2}{|\omega|^{1+\gamma}} d\omega \right)$$

or through an Itô stochastic integral formulation

$$W_c(t) = \frac{1}{\sqrt{\pi^2}} \int_0^t \frac{1}{|\omega|^{1+\gamma}} dW(\omega),$$

where $W(\omega)$ is a Wiener process, i.e., the usual Brownian motion, corresponding to $\gamma = 1/2$. Here, $c_0 = \sqrt{\Gamma(2+\gamma)/\Gamma(1+\gamma)}C_t$.

Filtering the integer samples of an fBm by

$$G_c(\omega) = \left( \frac{1}{\pi} \frac{1}{\omega + 2\pi}\right)^{-1}$$

gives a discrete white noise $\sim$ synthesis of an fBm by inverse filtering with $G_c^{-1}$.

**What is a fractional spline?**

If $0 < \alpha < 2$, any function $f(t)$ that can be written as $f(t) = \sum_{n \in \mathbb{Z}} a_n(t - n)^\alpha$ is a fractional spline of degree $\alpha$. A well-localized basis is the symmetric fractional B-spline $B^\alpha_k(t)$ characterized by the Fourier transform

$$\hat{B}^\alpha_k(\omega) = \left\{ \frac{\sin(\omega/2)^{\alpha+1}}{\omega/2} \right\}^k.$$

Example: if $\alpha = 1$, $B^1_k(t)$ is the triangle function

Many properties: valid $\alpha$-scale multi-resolution analysis, regularity, short “equivalent” support, fast interpolation algorithm (Fourier).

**Fractional spline interpolation**

$$f(n) \rightarrow f(\hat{n}) = B^\alpha_k(\hat{n}) \sum_{n \in \mathbb{Z}} f(n)B^\alpha_k(n-k),$$

where the fractional spline interpolation prefilters are defined as

$$B^\alpha_k(n) = \sum_{n \in \mathbb{Z}} (n-c_\alpha)^k = \sum_{n \in \mathbb{Z}} n^{k+\alpha}/\Gamma(\alpha+1).$$

Fast algorithm for the computation of $B^\alpha_k(n)$.

**What is the problem here?**

Find the optimal estimate of an fBm $W_c(t)$ from a series of noisy samples

$$y_k = W_c(k) + N(k),$$

where $N(k)$ is a Gaussian stationary with autocorrelation $r_k$ independent from $W_c$.

Formal Bayesian solution: $W_{\text{est}} = \mathcal{E} \{ W_c(t) | \{ y_k \}_{k \in \mathbb{Z}} \}$. The main result of this paper is the explicit computation of this solution.

A fractional spline estimate

The optimal estimate of a noisy fBm with Hurst exponent $\gamma$ is a fractional spline of degree $2\gamma$.

$$W_{\text{est}}(t) = \sum_{k \in \mathbb{Z}} c_k B^\gamma_k(t-k)$$

with $c_k = b_k + w_k - \lambda b_k = r_k$.

The constant $\lambda$ is chosen in such a way that $W_{\text{est}}(0) = 0$ and the Wiener-like filter is specified by

$$H(s) = \frac{1}{B^\gamma_k(s) + \left[ 2\sin(\frac{\pi}{e}\right]^{\alpha+1} R(s)^{\alpha/\alpha} s^\alpha}$$

Originality: since the fBm is not stationary, the usual Wiener-Hopf denoising filter solution does not apply here. However, an equivalent Wiener filter arises from the solution, followed by a non-stationary correction which tends to zero for large $t$.

**Conclusion**

The result produced here is essentially theoretical, but brings a renewed insight into the estimation of nonstationary processes:

- The optimal estimation space is built using shifts of the variogram (a similar result with more constraints on the estimation is known in Kriging approaches); here, this space is a fractional spline space;
- The best approximation of an fBm is non-stationary as well, but can be decomposed into the sum of a stationary part (filter), and of a short-lived correction;
- The estimation space inherits the scale invariance of the fBm, a property that provides efficient multi-resolution algorithms.

**Example: resampling a noisy fBm**

Given the noisy samples $y_k = W_c(k) + N(k)$, find the optimal estimate of $W_c(n/M)$ where $M \geq 2$ is an integer.

**Solution:** $W_{\text{est}}(n/M) = \sum_{k \in \mathbb{Z}} g^\gamma_{\text{est}}(k) B^\gamma_k(n-k)$

Implementation: using the scaling relation

$$B^\gamma_{\text{est}}(n/M) = \sum_{k \in \mathbb{Z}} g^\gamma_{\text{est}}(k) B^\gamma_k(n-k) = (g^\gamma_{\text{est}} * h^\gamma_k)[n]$$

and the FFT algorithm.

**Denoising result with \( \gamma = 0.6 \)**

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**Optimal Interpolation of a Fractional Brownian Motion Given its Noisy Samples**

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