On sampling lattices with similarity scaling relationships

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Abstract:
We provide a method for constructing regular sampling lattices in arbitrary dimensions together with an integer dilation matrix. Sub-sampling using this matrix leads to a similarity-transformed version of the lattice with a chosen density reduction. These lattices are interesting candidates for multi-dimensional wavelet constructions with a limited number of sub-bands.

1. Introduction to sampling lattices and related work

A sampling lattice is a set of points \( \{ R^k : k \in \mathbb{Z}^n \} \subset \mathbb{R}^n \) that is closed under addition and inversion. The non-singular generating matrix \( R \in \mathbb{R}^{n \times n} \) contains basis vectors in its columns. Lattice points are uniquely indexed by \( k \in \mathbb{Z}^n \) and the neighbourhoods around all sampling points are identical. This makes them suitable sampling patterns for the reconstruction of shift-invariant spaces.

Sub-sampling schemes for lattices are expressed in terms of a dilation matrix \( K \in \mathbb{Z}^{n \times n} \) forming a new lattice with generating matrix \( RK \). The reduction rate in sampling density corresponds to

\[
\text{det } K = \alpha^n = \delta \in \mathbb{Z}^+. 
\]

Dyadic sub-sampling discards every second sample along each of the \( n \) dimensions resulting in a \( \delta = 2^n \) reduction rate. To allow for fine-grained scale transitions we are particularly interested in low sub-sampling rates, such as \( \delta = 2 \) or 3.

As discussed by van de Ville et al. [8] the 2D quincunx sub-sampling is an interesting case permitting a two-scale relation. With the implicit assumption of only considering subsets of the Cartesian lattice it is shown that a similarity two-channel dilation may not extend for \( n > 2 \).

We show that by permitting more general basis vectors in \( \mathbb{R}^n \) the desired fixed-rate dilation becomes possible for any \( n \). Our construction produces a variety of lattices making it possible to include additional quality criteria into the search as they may be computed from the Voronoi cell of the lattice [9] including packing density and expected quadratic quantization error (second order moment). Agrell et al. [1] improve efficiency for the computation by extracting Voronoi relevant neighbours. Another sampling quality criterion appears in the work of Lu et al. [4] in form of an analytic alias-free sampling condition that is employed in a lattice search.

2. Lattice construction

We are looking for a non-singular lattice generating matrix \( R \) that, when sub-sampled by a dilation matrix \( K \) with reduction rate \( \delta = \alpha^n \), results in a similarity-transformed version of the same lattice, that is, it can be scaled and rotated by a matrix \( Q \) with \( Q^T Q = \alpha^2 I \). An illustration of a sub-sampling resulting in a rotation by \( \theta = \arccos \frac{1}{\sqrt{2}} \) is given in Figure 1. Formally, this kind of relationship can be expressed as

\[
QR = RK 
\]

leading to the observation that sub-sampling \( K \) and scaled rotation \( Q \) are related by a similarity transform

\[
R^{-1} QR = K. 
\]
Using a matrix \( J_2 = \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix} \) it is possible to diagonalize a 2D rotation matrix by the following similarity transform

\[
\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = J_2^{-1} \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix} J_2 = J_2^{-1} \Delta J_2. \tag{4}
\]

Using this observation to replace the scaled rotation matrix \( Q \) in Equation 3 leads to

\[
K = R^{-1}QR \\
\frac{K}{\alpha} = \det(R^{-1}J_n^{-1} \Delta S S^{-1}J_nR) = \alpha P \Delta P^{-1}
\]

with

\[
R = J_n^{-1}SP^{-1} \\
Q = \det(R^{-1}J_n^{-1} \Delta J_n).
\]

Thus, given a matrix \( K \) that has an eigen-decomposition corresponding to that of a uniformly scaled rotation matrix, we can compute the lattice generating matrix \( R \) as in Equation 6. The elements of the diagonal matrix \( S \) inserted in the construction of \( R \) scale the otherwise unit eigenvectors in the columns of \( P \). Below, we will refer to this construction as function \( \text{formRQ}(K, S) \) using \( S = I \) by default.

### 2.1 Constructing suitable dilation matrices \( K \)

The eigenvalues of \( K, \Delta \) and \( Q \) impose restrictions on their shared characteristic polynomial \( d(\lambda) = \det(K - \lambda I) = \sum_{k=0}^{n} c_k \lambda^k \) as discussed in the appendix. For the case \( n = \text{even} \) with the only non-zero coefficients \( c_0 = \delta, c_{n/2} < 4\delta, c_n = 1 \) this leaves a finite number of different options for \( c_{n/2} \). The case \( n = \text{odd} \) permits a single possible polynomial with non-zero coefficients \( c_0 = -\delta, c_n = 1 \). For these monic polynomials it is possible to directly construct a candidate \( K \) via the companion matrix (([6], p. 192))

\[
K = \begin{bmatrix} 0 & -c_0 \\ 1 & -c_1 \\ & \ddots & \ddots & \ddots \\ & & 1 & -c_{n-2} \\ & & & 1 & -c_{n-1} \end{bmatrix} \tag{7}
\]

This allows to construct a lattice fulfilling the self-similar sub-sampling condition for any dimensionality \( n \), one for every possible characteristic polynomial.

With this starting point it is possible to construct additional suitable dilation matrices via a similarity transform with a unimodular matrix \( T \)

\[
K_T = TK_T^{-1} = P_T \Delta P_T^{-1}. \tag{8}
\]

Using a unimodular rather than any non-singular \( T \) guarantees that \( T^{-1} \) is also unimodular following from the fact that \( T^{-1} \) can be constructed from the adjugate (the transpose co-factor matrix) of \( T \). Thus, \( K_T \) remains an integer matrix by this transform. Possible generators for this unimodular group are discussed in ([5], pp. 23). Our implementation, referred to as function \( \text{genUnimodular}(n) \), uses a construction of \( T = LU \) from several random integer lower and upper triangular matrices having ones on their diagonal.

It is not guaranteed that all possible \( K \) for a given characteristic polynomial can be generated through a similarity transform with some \( T \). However, \( \text{formRQ}(K_T) \) provides numerous non-equivalent \( R_T \) lattice generators. Among them it is possible to apply further criteria to select the “best” lattice.

An alternative to transforming \( K \) is the eigenvector scaling by diagonal matrix \( S \) in Equation 6. Using non-unit scaling allows to produce further lattices for any given \( K \) resulting in an \( n \)-dimensional continuous search space.

### 2.2 Construction Algorithm

The steps for constructing lattices with the desired sub-sampling matrices are summarized in Algorithm 1.

The function \( \text{compoly}(n, \alpha, C) \) is defined in the appendix. A possible implementation for the function \( \text{genUnimodular}(n) \) is described in Section 2.1 and \( \text{formRQ}(K) \) is defined near Equation 6. It should be noted that the list of lattices returned by \( \text{genLattices} \) may contain several equivalent copies of the same lattice. A Gram matrix implicitly represents angles between basis vectors as \( A = R^T R \). Two lattices \( R_1 \) and \( R_2 \), scaled to same determinant, are equivalent if their Gram matrices are related via \( A_1 = T^T A_2 T \) with a unimodular matrix \( T \in \mathbb{Z}^{n \times n} \) and \( \det(T) = 1 \). Determining this unimodular matrix is known to be a difficult problem, as it for instance also occurs when relating the adjacency matrices of two supposedly isomorphic graphs. Hence, our current method employs a simpler necessary test for equivalence by comparing the first few elements of the set
4. Discussion and potential applications

The current formation of candidate matrices $K$ based on similarity transforms of one valid example is not guaranteed to produce all possible solutions. For 2D and 3D we also employed an exhaustive search over a range of integer matrices with values in $[-3,3]$ resulting in the same number of non-equivalent 2D cases as the construction via $K_T$. However, for dimensionality $n > 3$ the exhaustive search had to be replaced by a random sampling of integer matrices ultimately rendering the method infeasible for $n > 5$. In that light the current construction via scaled eigenvectors of the companion matrix is a significant improvement as it allows to produce a large number of non-equivalent lattices for any dimensionality.

Our sub-sampling schemes may have applications for multi-dimensional wavelet transforms [7]. Another direction for possible investigation is the construction of sparse grids that are employed in the context of high-dimensional integration and approximation adapting to smoothness conditions of the underlying function space [3].
Appendix: Characteristic polynomial of a rotation matrix in $\mathbb{R}^n$

The similarity relationship between $K$ and $Q$ in Equation 2 implies that they share the same characteristic polynomial $d(\lambda) = \det(K - \lambda I) = \det(Q - \lambda I)$ leading to an agreement in eigenvalues $d(\lambda_k) = 0$ and determinant $d(0) \ (6)$, p. 184). Further, since $K$ is an integer matrix the polynomial $d(\lambda) \in \mathbb{Z}[\lambda]$ has integer coefficients $c_k$.

In order to find integer matrices $K$ with the eigenvalues of a scaled rotation matrix, it will be important to distinguish the two different forms of the diagonal matrix $\Delta$ in Equation 5 and 4 for the case $n = \text{even}$

$$\Delta = \text{diag}[e^{i \theta_1}, e^{-i \theta_1}, \ldots, e^{i \theta_{n/2}}, e^{-i \theta_{n/2}}]$$

and the case $n = \text{odd}$

$$\Delta = \text{diag}[1, e^{i \theta_1}, e^{-i \theta_1}, \ldots, e^{i \theta_{(n-1)/2}}, e^{-i \theta_{(n-1)/2}}]$$

with analogue block-wise constructions for $J_n$.

For dimensionality $n = \text{even}$ the characteristic polynomial fulfills

$$d(\lambda) = \prod_{k=1}^{n/2} (\alpha e^{i \theta_k} - \lambda)(\alpha e^{-i \theta_k} - \lambda)$$

$$= \prod_{k=1}^{n/2} (\alpha^2 - 2\alpha \cos \theta_k + \lambda^2)$$

$$= \prod_{k=1}^{n/2} \left( \frac{\alpha^4}{\lambda^2} - 2\alpha \lambda \cos \theta_k + \alpha^2 \right) \lambda^2$$

$$= d \left( \frac{\alpha^2}{\lambda} \right) \left( \frac{\lambda}{\alpha} \right)^n$$

The similarity relationship between $K$ and $Q$ in Equation 2 implies that they share the same characteristic polynomial $d(\lambda) = \det(K - \lambda I) = \det(Q - \lambda I)$ leading to an agreement in eigenvalues $d(\lambda_k) = 0$ and determinant $d(0) \ (6)$, p. 184). Further, since $K$ is an integer matrix the polynomial $d(\lambda) \in \mathbb{Z}[\lambda]$ has integer coefficients $c_k$.

Thus, if

$$d(\lambda) = \sum_{k=0}^{n} c_k \lambda^k$$

$$= \sum_{k=0}^{n} c_k \left( \frac{\alpha^2}{\lambda} \right)^k \left( \frac{\lambda}{\alpha} \right)^n$$

$$= \sum_{k=0}^{n} c_{n-k} \alpha^{n-2k} \lambda^k$$

$$\Leftrightarrow c_k = \alpha^{n-2k} c_{n-k} = C \delta^{1-2k} c_{n-k}.$$