ITM 1010

Computer and Communication Technologies

Lecture #13

Part II Introduction to Communication Technologies:

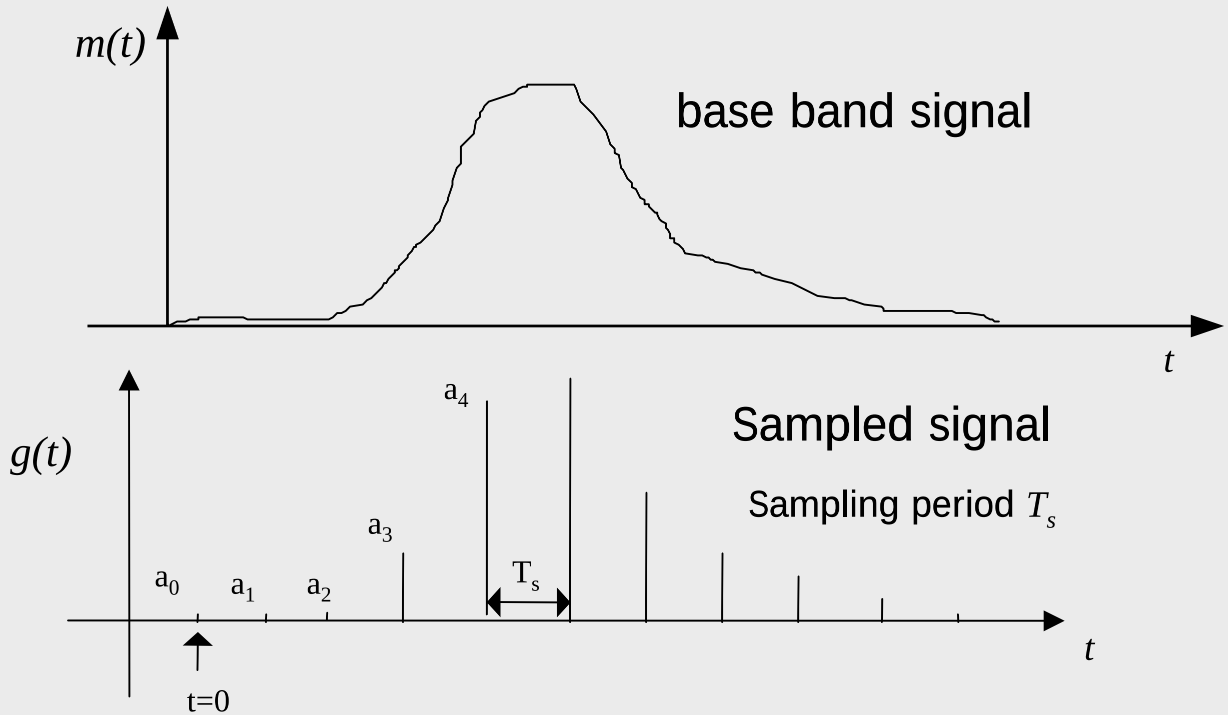
Digital Signals

Digital Signals: Sampling, Channel Capacity

What is the maximum error-free rate of information transmission in a communications channel?

Sampling Process

- Sampling is important first step to digitizing a continuous signal



- How many samples to fully reconstruct the original signal?
  - Must introduce Dirac delta functions and Discrete Fourier Transform to answer this question

Dirac-delta function

- Dirac- delta function $\delta(t)$ is defined by an impulse of unit area and infinitesimally short duration

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\delta(t) = \begin{cases} 0 & \text{for} \quad t \neq 0 \\ \infty & \text{for} \quad t = 0 \end{cases}$$

- Dirac-delta function has the useful property of “sifting” the value of a function which is multiplied by the Dirac-delta function in an integral:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$$

Discrete Fourier Transform

- A sampled signal consists of a sequence of *N* uniformly spaced pulses:

$$g(t) = a_0\delta(t) + a_1\delta(t - T_s) + a_2\delta(t - 2T_s) + \ldots + a_{N-1}\delta(t - (N-1)T_s)$$

$$g(t) = \sum_{n=0}^{N-1} m(nT_s)\,\delta(t - nT_s)$$

Original signal value at $t=nT_s$

Dirac-delta function at $t=nT_s$

- The spectrum of the ideal sampled signal may be found by taking the Fourier transform:

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\,e^{-j\omega t}dt \quad = \int_{-\infty}^{\infty} \sum_{n=0}^{N-1} m(nT_s)\,\delta(t - nT_s)\,e^{-j\omega t}dt$$

$$G(\omega) = \sum_{n=0}^{N-1} m(nT_s)\,e^{-j\omega nT_s}$$

- *G(ω)* is called the discrete Fourier transform

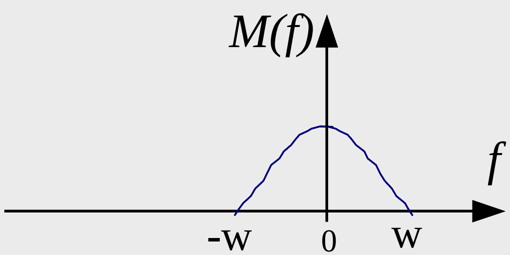
Properties of Discrete Fourier Transform

$$G(\omega) = \sum_{n=0}^{N-1} m(nT_s)\, e^{-j\omega n T_s}$$

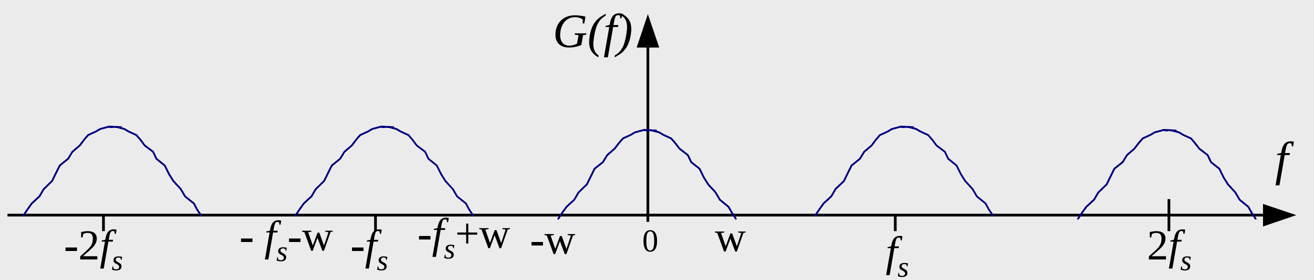
- $G(-\omega) = G^*(\omega)$; i.e negative frequencies is the phase conjugate (real part has same sign, imaginary part has opposite sign) of $G(\omega)$.

- $G(\omega)$ is formed by a summation of sinusoids. Hence it is periodic. And the spectrum of sampled signal repeats every f_s, where $f_s = 1/T_s$ is the sampling frequency

- Suppose the base band signal $m(t)$ is band-limited to a bandwidth of w If we sample at a rate $f_s > 2w$ the sampled signal spectrum will look like:



Spectrum of continuous signal $m(t)$



Spectrum of sampled signal $g(t)$

Nyquist's Criterion

- The minimum sampling rate needed to avoid loss of information in a signal which is band-limited to a bandwidth of w is $2w$.

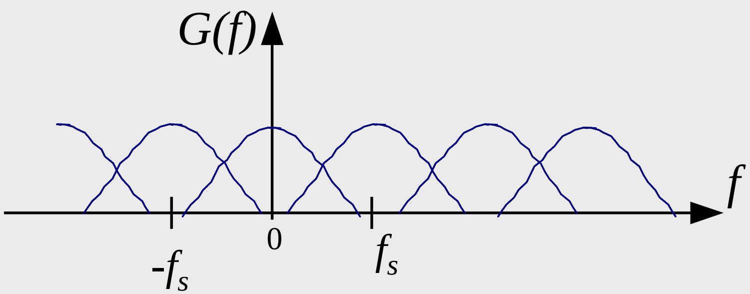
Nyquist's criterion: $f_s \geq 2w$

- A band-limited signal can be completely represented by its sampled waveform if the sampling rate satisfies the Nyquist criterion: the original signal may be recovered with no loss of information whatsoever from the sample values.

- In practical systems, the sampling rate must be slightly faster than the Nyquist rate to allow rejection of the adjacent spectrum using practical filters.

Aliasing

- If the sampling rate is less than the Nyquist rate, the original signal spectrum cannot be recovered from the sampled data spectrum because of overlap with adjacent periods of the sampled spectrum.



Aliasing when $f_s < 2w$

- Aliasing gives rise to severe signal distortion. The amplitude of high frequency components of the original signal appears in the low frequency part of adjacent spectra.
- Low-pass filters are used to limit the bandwidth of the original base band signal if the sampling rate is low in order to prevent aliasing.

Communications channel capacity

- We are now in a position to consider the question "Given a communications channel of bandwidth B hertz, what is the maximum rate at which information may be sent through?"

- Channel can carry an analog signal of bandwidth B

- The analog signal may be completely represented by samples made every 1/(2B) seconds (the Nyquist sampling rate). The channel may therefore carry 2B samples per second

- In the absence of noise, each sample may have an infinite number of different levels so the noise-free channel can send an infinite number of bits per second.

- But in practice all communications channels have noise

Detecting different signal levels in noise

- Simplest type of noise is white noise (Gaussian noise) which is characterized by:
  - Flat frequency spectrum over the bandwidth of the signal
  - Amplitude of the noise follows a Gaussian distribution
  - May be completely specified by giving its mean square amplitude N (which is its power in a unit resistance)

- If received signal power from a noisy communications channel is P, the number of different amplitude levels that may be distinguished in the received signal in the presence of noise with mean square amplitude N is of the order

$$\text{Number of levels} \approx \sqrt{\frac{P+N}{N}}$$

Shannon's Channel Capacity Theorem

- In 1948 Claude Shannon derived the following formula for the capacity C (defined as the maximum rate at which information can be sent without error) in a channel of bandwidth B and subject to white noise of mean power N :

$$C = 2B\log_2\sqrt{\frac{P+N}{N}} \quad \text{bits per second}$$

$$\Rightarrow \quad C = B\log_2(1+P/N) \quad \text{bits per second}$$

- P/N is signal-to-noise ratio in received signal

- Shannon's Theorem (Fundamental theorem of information theory): Given a discrete memory-less channel with capacity C and a source with rate R<C, there exists a code such that the output of the source can be transmitted over the channel with arbitrarily small probability of error.

- capacity of a channel may be increased by increasing P/N

Decibels (dB)

- The signal to noise power ratio is usually expressed in decibels

$$\text{signal to noise ratio (in dB)} = 10\log_{10}\left(\frac{P}{N}\right)$$

- Any ratio given in dB must be a power ratio

- Example: What is the channel capacity of a voice telephone line if it has a 32dB signal to noise ratio and a bandwidth of 4KHz?
- Solution:

$$\text{signal to noise ratio (in dB)} = 10\log_{10}\left(\frac{P}{N}\right) = 32$$

$$\frac{P}{N} = 10^{3.2} = 1584.9$$

$$C = B\log_2(1 + P/N) = 4000\log_2(1585.9)$$

$$C = 4000\frac{\log_{10}(1585.9)}{\log_{10} 2} = 42524$$ bits per second

N.B. 4KHz channel bandwidth however limits transmission rate < 8000 symbols per second.

Summary

- Nyquist sampling rate: An analog signal with a finite bandwidth w may be completely specified by sampling at a rate 2w.
  - Sampling at less than 2w will produce aliasing errors

- Shannon's Channel Capacity Theorem: The maximum rate at which information may be sent in a channel which has white Gaussian noise is

$$C = B\log_2\left(1 + P/N\right) \quad \text{bits per second}$$