Synthetic asymptote formula for surface-printed resistor

S.-K. Kwok, K.-F. Tsang, Y.L. Chow and K.-L. Wu

Abstract: To date, the design of surface-printed resistors has utilised the time-consuming moment method. The synthetic asymptote technique usually generates simple design formulae, and this technique has been extended and applied to the design of surface-printed resistors. It is found that the derived resistance is immune to change even at high frequencies. In addition, it is found that the novel formula for resistance complies with the Pythagorean nature of trigonometry, which forms a simple basis for synthesis and analysis. In comparison with the moment method solution, it is found that the resistance formula has an error of about 3% for all practical slab dimensions. The synthetic asymptote developed allows designers to rapidly determine the resistance.

1 Introduction

The simplest circuit element in electronics is a resistor, and in a low temperature co-fired ceramic (LTCC) circuit, it is likely to be surface printed. A surface-printed resistor [1] is a slab of rectangular cross-section, e.g. in a horizontal position, conveniently printed with its ends on top of its two horizontal electrodes, as shown in Fig. 1*a*. In such structure, the current flow in the resistor must turn vertically at the ends before entering the electrodes, as shown in Fig. 1*b*. As a result of this change in direction of current flow, estimation of the value of resistance is difficult.

Using a synthetic asymptote [2–5], this paper demonstrates that the estimation of resistance is not a difficult task and that a sufficiently accurate reistor value can be obtained.

Basic assumptions are stated as follows:

(a) The printed resistor is small in terms of the operating wavelength in free space. The resistivity of the resistor is normally high so that the skin effect can be ignored. As a result, the current flow distribution inside the resistor satisfies the Laplace equation (see Fig. 1*b*) and is analogous to the static electric flux.

(b) There is no current flow outside the surface of the resistor. This means that, except at the electrodes, the surface presents an impermeable wall to the current flow (analogous to a magnetic wall to the electric flux).

(c) The impermeable walls, opposing current flow in the *w*-direction in Fig. 1*a*, imply that there is no variation in current distribution in the *w*-direction. Consequently, the problem is reduced to static two-dimensional (2D) current flow on the x–y plane.

Owing to the fact that there is chemical reaction between the resistor ink and the unfired LTCC tape, it is currently difficult to control the exact composition and profile of a printed resistor. Hence, a comparison between calculated and experimental values is impractical. To this end, the derived formula will be compared with the moment method solution [6] only.

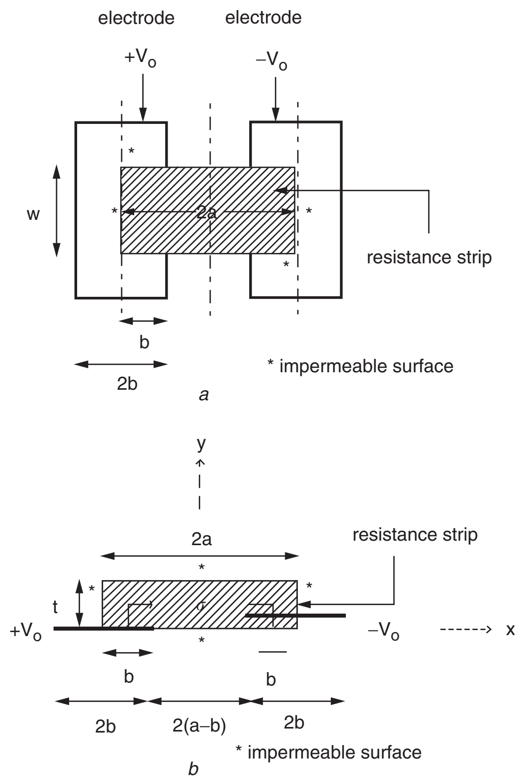

Fig. 1 *Surface-printed resistor slab*
a Top view of surface-printed resistor slab soldered to electrodes. Slab length 2*a*, width *w*, thickness *t*; soldered over width *b* on each electrode
b Cross-sectional view of the resistor (from one electrode to the other)

2 CAD formula employing the synthetic asymptote

The synthetic asymptote is constructed from two regular asymptotes; namely, the near asymptote when the slab

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thickness t approaches zero (see Fig. 1*b*), and the far asymptote when the slab thickness t approaches infinity (see Fig. 2*a*). As will be discussed, the synthetic asymptote of t is also the synthetic asymptote of the slab length a, but in the reciprocal sense; that is, the near asymptote of t becomes the far asymptote of a, and vice versa. This exceptional synthetic asymptote of dual variables results in a simple and accurate resistance formula for the printed resistor.

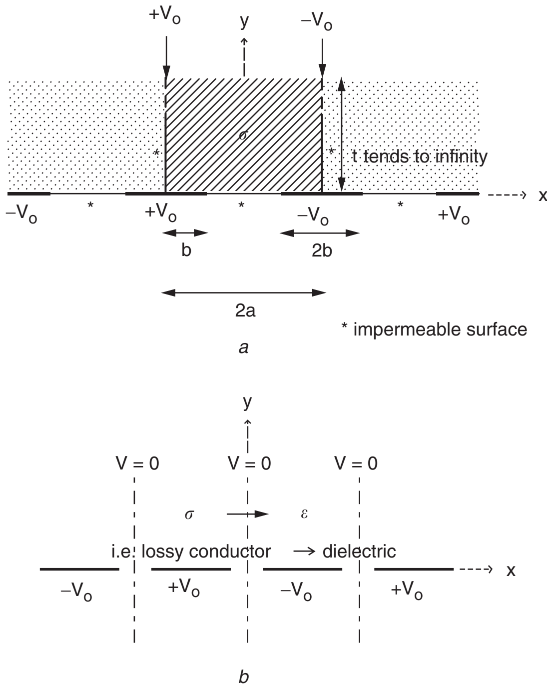

Fig. 2 *Cross-sectional view of resistance slab and multiple reflections (images) along x-direction*
a Cross-sectional view of a resistance slab with infinite thickness
b Images along x-direction, from side and bottom (impermeable) walls (*)

2.1 Thin slab (near) asymptote and equivalent long slab (far) asymptote

When the slab is very thin, the main part of the current flows along the horizontal portion of the slab between the two electrodes (Fig. 1*a* and 1*b*). The resistance R_α of the slab (of conductivity σ) is simply:

$$R_\alpha(t) = \lim_{t \to 0} R = \frac{2}{\sigma w}\frac{(a-b)}{t} \tag{1}$$

where w is the width of the slab. It should be noted that (1) is describing a 3D scenario (with impermeable walls).

When t is invariant (in the denominator in (1)) and electrode width b is constant, it is seen that (1) becomes the *far* asymptote of a resistor of length $2(a-b)$. This dual asymptote property, near and far, of t and a respectively in (1) will be useful in the derivation of the synthetic asymptote and the following illustrative numerical examples.

2.2 Thick slab (far) asymptote and equivalent: short slab (near) asymptote

When the slab is very thick, the impermeable wall at the top in Fig. 2*a* extends to infinity. However, the remaining three impermeable walls yield multiple reflections (images) along the x-direction of the electrodes and result in the structure shown in Fig. 2*b*. In contrast, multiple images along the y-direction from the top and bottom walls have little impact. If the medium around the electrodes is a dielectric, following Collin's analysis [7], there exists a conformal mapping solution for the capacitance between the two adjacent electrodes. By making use of the duality between capacitance and conductance, the resistance formula is obtained as follows:

$$R_\beta = \lim_{t \to \infty} R = \frac{2}{\sigma w}\frac{K(k')}{K(k)} \tag{2}$$

where

$$k = \sin\frac{\pi b}{2a}, \quad k' = \sqrt{1-k^2}$$

and $K(k)$ is the complete elliptic integral of the first kind. Similar to (1), here w is used again to obtain the 3D solution. Unlike the near asymptote in (1), the far asymptote in (2) is actually independent of the thickness t (as long as t is large).

If t is small (note that the resistor length $2a$ is much smaller), (2) is still accurate and becomes the near asymptote of a. This second dual asymptote property of (2) (of t and a) is complementary to that in (1).

2.3 Synthetic asymptote — the CAD formula

A synthetic asymptote is an equation R constructed to curve-fit between both regular asymptotes, that is R_α and R_β in this paper. Our objective is to employ the simplest curve-fit that gives acceptable results. In view of the trigonometric nature of the problem, the following form is used:

$$R(t) = (R_\alpha^p + R_\beta^p)^{1/p} \tag{3}$$

where p is a variable to be optimised

It will be demonstrated in the numerical examples in Section 3 that the power p in (3) may simply be chosen as $p = 2$ for *all* possible values (from zero to infinity) of both resistor length $2a$ and thickness t.

3 Numerical verification of the CAD formula

With the self and mutual coefficients obtained, the moment method employing the Green's function shown in the Appendix (Section 7) is now applied to the 2D structure in Fig. 1*b*. Each electrode b is divided into 80 equal segments. The length b is 8 mils and the width w is 20 mils (mil = 0.001 inch). A typical value of conductivity is chosen; namely, $\sigma = 1000$ S/m. With the electrode b remaining constant, the thickness t and length $2a$ are varied, in turn, resulting in four different resistor structures.

3.1 Example 1 (Fig. 3)

Condition: Resistor half-length $a = 50$ mils (referred as *long*), t varies.

Figure 3 compares the curve of the moment method solution with curves of the synthetic asymptote from (3). To investigate convergence of the synthetic asymptote, the near and far asymptotes using (1) and (2) are plotted. Equations (1) and (2) are further summed to produce the fine continuous line of (3) with $p = 1$. This fine line does not agree well with the moment method solution. To proceed, the power p is further matched by attempting more numerical trials so that (3) matches the moment method solution at one point. Without loss of generality, the point around $t = 14$ mils is chosen and thus the power $p = 2$ is obtained. Now, (3) with $p = 2$ matches the moment method solution very well for all t. The error incurred is about ± 0.3 ohm.

It is found that the selection of the intermediate matching points (at t and a) are not critical in the determination of the power p, nor is the exact value of p critical in keeping the

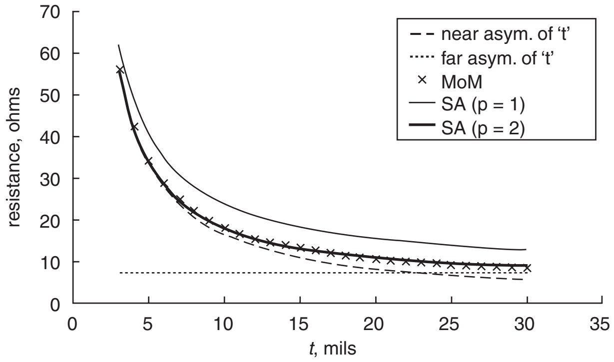

Fig. 3 *Resistance R against slab thickness t, with resistor half-length a = 50 mils: by the moment method, by the asymptotes of upper and lower limits, and by the CAD formula of synthetic asymptote equation (3) with power p = 1 and 2*
Resistor width $w = 20$ mils with conductivity of 1000 S/m

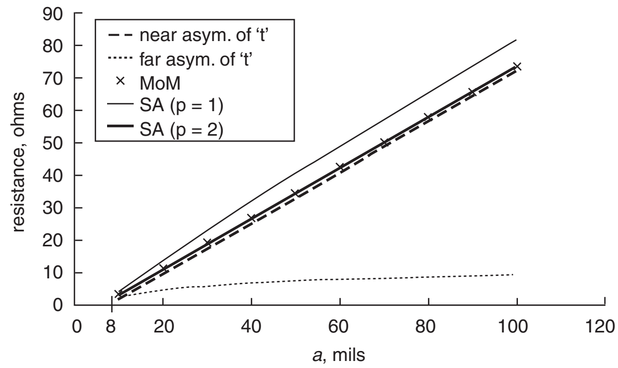

Fig. 5 *Resistance R against resistor length 2a, with resistor thickness t = 5 mils: by the moment method, by the asymptotes of upper and lower limits, and by the CAD formula of synthetic asymptote equation (3) with power p = 1 and 2*

error small. This is understandable as the two asymptotic ends of the curve in Fig. 3 have already been accurately pegged by (1) and (2). In order to investigate the accuracy, a second example is given below.

3.2 Example 2 (Fig. 4)

Condition: Resistor of half-length $a = 20$ mils (referred to as *short*), t varies.

Figure 4 shows a comparison of the resistance curves using (3) and the moment method solution. It is seen that the results agree well. Figs. 3 and 4 together clearly demonstrate that the formula (3) applies equally well to both long and short resistors, and in fact to all resistor lengths since it is a synthetic asymptote of the resistor length $2a$.

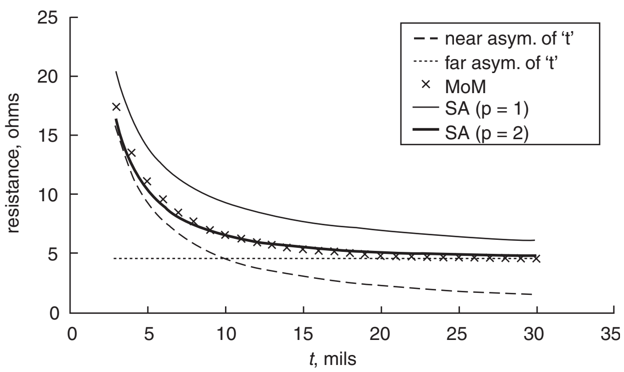

Fig. 4 *Resistance R against slab thickness t, with resistor half-length a = 20 mils: by the moment method, by the asymptotes of upper and lower limits, and by the CAD formula of synthetic asymptote equation (3) with power p = 1 and 2*

3.3 Example 3 (Fig. 5)

Condition: a varies, $t = 5$ mils.

As shown in Fig. 5, (3) is plotted. In this case, the near and far asymptotes and the synthetic asymptote with $p = 1$ and $p = 2$ are also plotted. Again, it is seen that, with $p = 2$, the synthetic asymptote curves agree well with those of the moment method solution. The error amplitude is about ± 0.3 ohm in general.

3.4 Example 4 (Fig. 6)

Condition: a varies, $t = 20$ mils

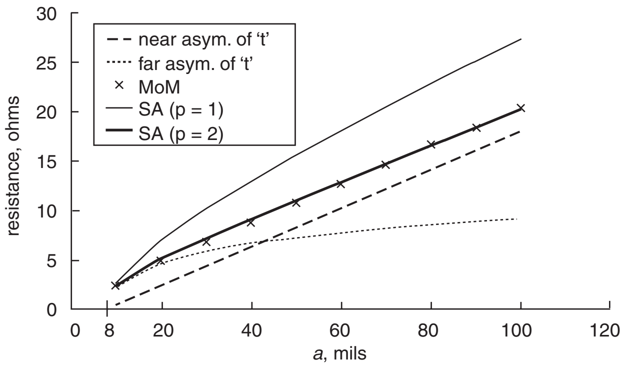

Fig. 6 *Resistance R against resistor length 2a, with resistor thickness t = 20 mils: by the moment method, by the asymptotes of upper and lower limits, and by the CAD formula of synthetic asymptote equation (3) with power p = 1 and 2*

In such circumstance, the error amplitude is again ± 0.3 ohm. With t held constant, (1), being the near asymptote of t, becomes the far asymptote of the resistor with half-length a. Moreover, the far asymptote of t, becomes the near asymptote of the resistor with half-length a (i.e. (2)). Figs. 5 and 6 again demonstrate that (3) applies equally well for all resistor thicknesses.

To summarise, the behaviour of resistance values with varying a and t has been analysed. The analysis and comparisons are shown in Figs. 3–6. An engineer gains a rapid insight into the design as a result of the simplicity of the asymptotes (1) and (2) and the synthetic asymptote (3).

It is also concluded that by choosing $p = 2$, in (3), the solution is generally accurate for all resistor dimensions (thickness t, length $2a$ and soldered width b on the electrode) with an error of $\sim 3\%$ (corresponding to an error amplitude of ± 0.3 ohm).

In order to provide better insight, the following investigations were made:

(a) very closely spaced electrodes, i.e., $(a - b) \ll b$ (see Fig. 1*b*); and

(b) very thin resistor slab, i.e. $t \ll b$.

It is seen in Fig. 4 that the maximum error between (3) and the moment method solution can reach 6% at $t = 3$ mils. In that case, the corresponding resistance is low, about 16 ohms. At such low resistance (Fig. 4), the maximum error is bounded by the far asymptote of t. On the other hand, when $t \ll b$, the maximum error is bounded by the near asymptotic of (1). Evidently, near this limit, (3) cannot have much error. In contrast, it is clear that the error incurred (6%) comes from the moment method solution due to the strong coupling fields. Thus, it is seen that in such circumstances, (3) is more accurate than the moment method solution. Incorporating the former analysis, the error incurred is $\sim 3\%$ in general.

4 Conclusions

A resistance formula for printed resistors has been derived. There is little skin effect, so that this resistance is immune to change even at higher frequencies. The Pythagorean nature of the resistance (3) has been demonstrated.

In fact, (3) is a formula of 5 independent variables which is obtained by one numerical match of the power p. It is found that this novel formula has an average error of 3% for all practical slab dimensions. The reason for the accuracy and simplicity is that the derived synthetic asymptote depends on two variables, the thickness t and the length a of the substrate, instead of the normal case of one variable.

It should be noted that the derived formula cannot be verified with available empirical design rules [8]. This is due to the intrinsic inconsistent measurement of the fabricated resistor on LTCC. Up to this point in time, there is no reliable measured data for comparison. In contrast with the experimental difficulty, the synthetic asymptote developed allows designers to acquire fast and efficient insight into the resistance.

5 Acknowledgments

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7 Appendix: Special Green's function for the moment method

To verify (3), no commercial software is available that gives the boundary condition of the impermeable current walls of the resistor. Hence, in order to obtain independent numerical results, a Green's function is developed for the moment method in this Appendix.

The periodic structure of impermeable walls along the x-direction is shown in Fig. 2b. By the 2D classical variable separation [7], with a field point x and a point source x' on the electrode at $y'=0$, the Green's function is

$$G_0 = \sum_{n=1,3,\ldots}^{\infty} \frac{2}{n\pi} \cos\frac{n\pi x}{2a} \cos\frac{n\pi x'}{2a} e^{\mp(n\pi y/2a)} \tag{4}$$

Note that the Green's function has exponential decay along both $\pm y$ directions.

The Green's function assumes no permeable walls along the y-direction. If impermeable walls are assumed along the y-direction of the soldered resistor in Fig. 1b, there are periodic multiple images at separations of $2t$ along y. Each image has a field given by (4) in Fig. 2a. Because of the exponential decay, these fields can easily be summed, giving the potential Green's function G, and the potential coefficients of the moment method. The derivation is shown below.

The mutual (potential) coefficient P_{ij} in the matrix of the moment method

$$(V_i) = \frac{1}{\sigma \Delta x'} [P_{ij}](I_j)$$

between the field point i at x and the source point j at x', is simply the potential Green's function. The result is

$$P_{ij} = G(x,x') = \sum_{n=1,3,\ldots}^{\infty} \frac{2}{n\pi} \cos\frac{n\pi x}{2a} \cos\frac{n\pi x'}{2a} \sum_{m=1}^{\infty} 2e^{-m(n\pi t/a)} + \sum_{n=1,3,\ldots}^{\infty} \frac{2}{n\pi} \cos\frac{n\pi x}{2a} \cos\frac{n\pi x'}{2a}$$

where this first (double) sum is the potentials of the images above and below the x-axis, and the second sum is the source and images along the x-axis. The first and second terms may be grouped together as

$$P_{ij} = \sum_{n=1,3,\ldots}^{\infty} \frac{2}{n\pi} \cos\frac{n\pi x}{2a} \cos\frac{n\pi x'}{2a} \left[\sum_{m=1}^{\infty} 2e^{-m(n\pi t/a)} + 1 \right]$$

The mutual coefficient is then reduced to

$$P_{ij} = G(x,x') = \sum_{\pi=1,3,\ldots}^{\infty} \left[\left(\frac{2}{n\pi} \cos\frac{n\pi x}{2a} \cos\frac{n\pi x'}{2a} \right) \left(\frac{1+e^{-n\pi t/a}}{1-e^{-n\pi t/a}} \right) \right] \tag{5}$$

The self-coefficient P_{ii} of potential is now derived. Using conformal mapping, a 2D segment of width Δx is equivalent to a cylinder of radius $\Delta x/4$. Also, the self-coefficient on a cylinder is the potential Green's function on the surface of the cylinder, from a unit charge at the centre of the cylinder. It is then easy to see that the self term of a segment i at x is the average of the Green's functions of (5) at $\pm\Delta x/4$ from x, i.e.

$$P_{ii} = \frac{1}{2}[G(x, x-\Delta x/4) + G(x, x+\Delta x/4)] \tag{6}$$

With the self and mutual coefficients found, the matrix of the moment method is solved in order to obtain the resistance R of the resistor shown in Fig. 1. To ease numerical computation of the coefficients, (5) and (6) are truncated at the $(n=)$ 199th term. Such a large number of terms is used to ensure high accuracy in the moment method before the results are compared with the derived formula (3).