# A Plane Wave Expansion of Spherical Wave Functions for Modal Analysis of Guided Wave Structures and Scatterers

Robert H. MacPhie, *Life Fellow, IEEE*, and Ke-Li Wu, *Senior Member, IEEE*

***Abstract*—A new finite plane wave series expansion for spherical wave functions of the first kind is presented in this paper. The formulation converts the spherical wave function described in the spherical coordinate system into a series of plane wave functions represented in the Cartesian coordinate system. The series expansion will be very useful in modal analysis of three dimensional guided wave structures and scatterers containing planar boundary surfaces. For a given range of orders $m$ and degree $n$ and for a region with $|\vec{r}| < R$, the *same set* of plane waves can be used. The theory is numerically verified for a wide range of parameters, showing its fast convergence characteristics. The plane wave expansions of the vector multipole fields can also be obtained.**

***Index Terms*—Electromagnetic theory, guided waves, modal analysis, scattering, waveguide.**

## I. Introduction

SPHERICAL harmonic wave functions are canonical solutions of the Helmholtz equation in spherical coordinates. They have been well-understood [1]–[3] and widely used in analyzing scattering problems as well as microwave devices consisting of curved spherical surfaces.

In solving a practical electromagnetic problem one must frequently resort to some form of series expansion. The nature of the expansion basis functions will be governed by the size of the object with respect to the wavelength of interest and the boundary shapes. The sensible choice for the basis functions is to use the wave harmonic modal functions in the coordinate system that best fits the boundary shapes. This is the way in which Mie considered the diffraction of a plane wave by a sphere [4] using spherical harmonic wave functions. However, in most practical problems, it is not possible to restrict the boundary of a scatterer to a single coordinate system and one often uses numerical techniques to solve such problems.

Electromagnetic modal analysis has been widely used in solving guided wave problems. The solutions provided by the modal analysis are analytical and therefore are more efficient and accurate than numerical techniques. Recently, the authors of this paper proposed the theory of the finite plane-wave series expansion, in which circular cylindrical wave basis functions were expanded as a finite series of plane wave functions. The theory has been used for solving many guided wave problems such as arbitrarily shaped E- and H-plane waveguide two dimensional (2-D) discontinuities consisting of piecewise planar and cylindrical boundaries without the need of numerical integration [5]–[7].

A question arises when an attempt is made to solve a 3-D guided wave problem using modal analysis, such as the modal analysis of a waveguide taper or an irregularly shaped cavity: Is there a plane wave series expansion for the spherical wave harmonic functions? In this paper we present the general theory for converting a spherical wave harmonic function of the first kind into a rapidly converging series of plane wave functions. This double series is the discretization of the double integral representation given by Stratton [2, p. 410] and which is much more convenient in the numerical solution of scattering problems, e.g., the plane wave scattering by a perfectly conducting cube [8] and scattering of a double plane waveguide taper. Having had the plane wave series expansion, the electromagnetic fields on a piecewise planar boundary of a 3-D inhomogeneous region can be expressed as a series of plane wave functions which will greatly facilitate the modal analysis of such problems and of the scattering from 3-D surfaces with planar boundaries.

## II. Theoretical Development

In this section, we will deduce the transformation from the scalar spherical wave function

$$\Psi_{nm}(r,\,\theta,\,\phi) = j_n(kr)P_n^m(\cos\theta)e^{jm\phi} \tag{1}$$

to a series of plane wave functions, where $j_n(x)$ is the spherical Bessel function of the first kind and $P_n^m(x)$ is the associated Legendre function.

It is well known that a plane wave propagating in the direction

$$\hat{k} = \hat{x}\sin\alpha\cos\beta + \hat{y}\sin\alpha\sin\beta + \hat{z}\cos\alpha \tag{2}$$

can be expanded in terms of spherical wave functions as [2]

$$e^{-j\vec{k}\cdot\vec{r}} = \sum_{n=0}^{\infty}(-j)^n(2n+1)j_n(kr) \cdot \left\{ \begin{array}{l} P_n(\cos\alpha)P_n(\cos\theta) \\ +2\sum_{m=1}^{n}\dfrac{(n-m)!}{(n+m)!}P_n^m(\cos\alpha) \\ \cdot P_n^m(\cos\theta)\cos(m(\phi-\beta)) \end{array} \right\} \tag{3}$$

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R. H. MacPhie is with the Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, ON, Canada N2L 3G1.

K.-L. Wu is with the Department of Electronic Engineering, The Chinese University of Hong Kong, Shatin, NT, Hong Kong, PRC.

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where the observation position vector is defined by

$$\vec{r} = r\left[\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta\right] = \hat{x}x + \hat{y}y + \hat{z}z \tag{4}$$

Since $\cos(m(\phi-\beta)) = (1/2)[e^{jm\phi - jm\beta} + e^{-jm\phi + jm\beta}]$, it is straightforward to show that (3) can be rewritten as

$$e^{-j\vec{k}\cdot\vec{r}} = \sum_{n=0}^{\infty}(-j)^n(2n+1)j_n(kr)\sum_{m=-n}^{n}\frac{(n-|m|)!}{(n+|m|)!} \cdot \left[P_n^{|m|}(\cos\theta)e^{jm\phi}\right]\left[P_n^{|m|}(\cos\alpha)e^{jm\beta}\right]. \tag{5}$$

Using (1) in (5) we have

$$e^{-j[k_x(\alpha,\beta)x + k_y(\alpha,\beta)y + k_z(\alpha,\beta)z]} = \sum_{n=0}^{\overline{N}}\sum_{m=-n}^{n} M_{nm}(\alpha,\beta)\overline{\Psi}_{nm}(r,\theta,\phi) \tag{6}$$

where

$$M_{nm}(\alpha,\beta) = (-j)^n(2n+1)\frac{(n-|m|)!}{(n+|m|)!}\left[P_n^{|m|}(\cos\alpha)e^{-jm\beta}\right] \tag{7}$$

and

$$\overline{\Psi}_{nm}(r,\theta,\phi) = j_n(kr)P_n^{|m|}(\cos\theta)e^{jm\phi}. \tag{8}$$

In (6), the fact that $j_n(kr) \approx 0$ when $kr < \overline{N} - N_0$, where $N_0$ is a small integer relaxation constant for a given range of $n$, $m$ and $r$, has been used to truncate the infinite series. $\overline{N}$ can be chosen such that $\overline{N} = \{kR\}_I + N_0$, where $R$ is the maximum value in the range of $r$ and $\{\alpha\}_I$ is the first integer greater than $\alpha$. Strictly speaking, since $P_n^{-m} = (-1)^m((n-m)!/(n+m)!)P_n^m$, the absolute value of $m$ in (7) is not necessary. However, it may provide some convenience in programming.

As illustrated in Fig. 1, the following relation holds:

$$\sum_{n=0}^{\overline{N}}\sum_{m=-n}^{n}(\cdot) = \sum_{m=-\overline{N}}^{\overline{N}}\sum_{n=|m|}^{\overline{N}}(\cdot). \tag{9}$$

Thus, (6) can be expressed as

$$\sum_{m=-\overline{N}}^{\overline{N}}\sum_{n=|m|}^{\overline{N}}(-j)^n(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\cos\alpha) \cdot \left\{j_n(kr)P_n^m(\cos\theta)e^{jm\phi}\right\}e^{-jm\beta} = e^{-j\vec{k}\cdot\vec{r}}. \tag{10}$$

If we denote

$$R_n^m(\alpha) = (-j)^n(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\cos\alpha) \tag{11}$$

(10) becomes

$$\sum_{m=-\overline{N}}^{\overline{N}}\sum_{n=|m|}^{\overline{N}} R_n^m(\alpha)\cdot\Psi_{nm}(\vec{r})e^{-jm\beta} = e^{-j\vec{k}(\alpha,\beta)\cdot\vec{r}} \tag{12}$$

where $\Psi_{nm}(\vec{r})$ is given by (1).

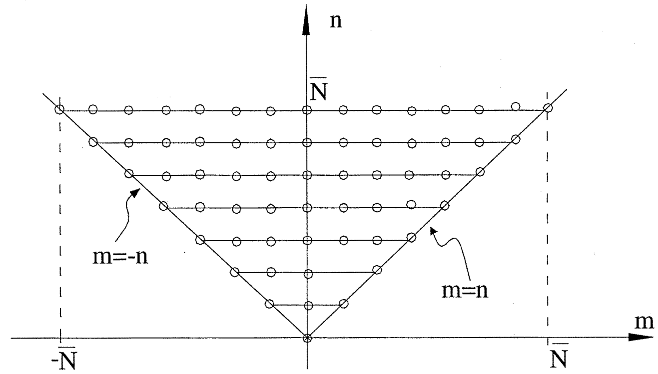


Fig. 1. Indices diagram of the double summation in (6).

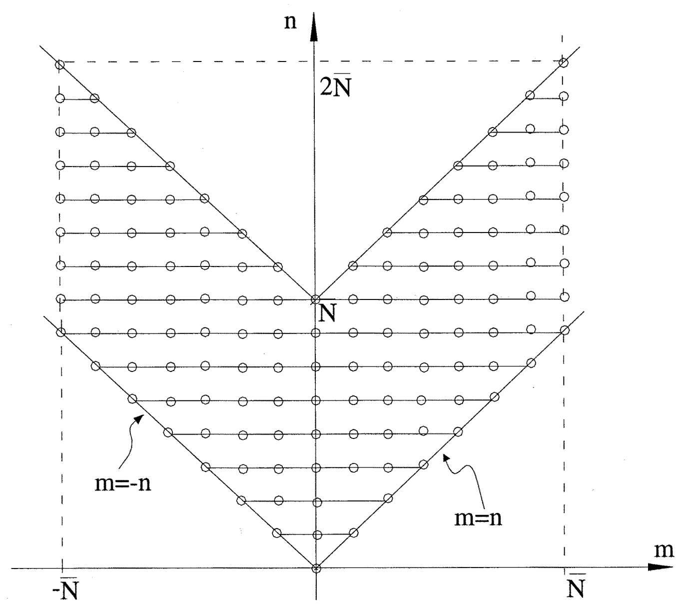


Fig. 2. Modified indices diagram of the double summation in (13).

Since it has been assumed that $j_n(z) = 0$ when $n > \overline{N}$, we can modify (12) as follows:

$$\sum_{m=-\overline{N}}^{\overline{N}}\sum_{n=|m|}^{\overline{N}+|m|} R_n^m(\alpha)\cdot\Psi_{nm}(\vec{r})e^{-jm\beta} = e^{-j\vec{k}(\alpha,\beta)\cdot\vec{r}} \tag{13}$$

The major advantage of (13) over (12) is that, for any $\mathrm{m}$, the number of terms in the series on $\mathrm{n}$ of (13) is a *constant* and equal to $\overline{N}+1$. It is this modification that enables any spherical wave function $\Psi_{nm}(\vec{r})$ to be expanded by a series of *the same set* of plane wave functions, regardless of the vector $\vec{r}$. In fact, the extra terms added in the inner summation from $n = \overline{N}+1$ to $n = \overline{N}+|m|$ will make approximation (13) more accurate than its original form (12).

The graphical representation of the modification is given in Fig. 2. As mentioned above, all terms for $\overline{N} < n$ are negligible, but are included in the summation of (13).

If we define

$$\tilde{Q}_m(\alpha,\vec{r}) = \sum_{n=|m|}^{\overline{N}+|m|} R_n^m(\alpha)\Psi_{nm}(\vec{r}) \tag{14}$$

and set $N = 2\overline{N} + 1$ and $q = m + \overline{N}$, (13) can be uniformly sampled at $N$ discrete azimuth angles $\beta$ as

$$\frac{1}{N}\sum_{q=0}^{N-1}\underbrace{\tilde{Q}_{q-\overline{N}}(\alpha,\,\vec{r})}_{Q_q(\alpha,\,\vec{r})}e^{-jql(2\pi/N)} = \underbrace{\frac{e^{-j\overline{N}\beta_l}}{N}\,e^{-j\vec{k}(\alpha,\beta_l)\cdot\vec{r}}}_{W_l(\alpha,\,\vec{r})} \tag{15}$$

where $\beta_l = l(2\pi/N)$. It can be observed that (15) is the *discrete Fourier transform (DFT)* [9, pp. 358–359] of $Q_q(\alpha,\,\vec{r})$. Therefore, the inverse of the transform gives

$$Q_q(\alpha,\,\vec{r}) = \sum_{l=0}^{N-1} W_l(\alpha,\,\vec{r})e^{jql(2\pi/N)}. \tag{16}$$

But

$$Q_q(\alpha,\,\vec{r}) = \tilde{Q}_{q-\overline{N}}(\alpha,\,\vec{r}) = \sum_{n=|q-\overline{N}|}^{\overline{N}+|q-\overline{N}|} R_n^{q-\overline{N}}(\alpha)\Psi_{n,\,q-\overline{N}}(\vec{r}). \tag{17}$$

By recalling that $m = q - \overline{N}$, we obtain

$$\sum_{n=|m|}^{\overline{N}+|m|} R_n^m(\alpha)\Psi_{nm}(\vec{r}) = \sum_{l=0}^{N-1} W_l(\alpha,\vec{r})e^{j(\overline{N}l2\pi/N)}e^{jml(2\pi/N)}. \tag{18}$$

The next task is to obtain the spherical wave function $\Psi_{n,\,m}(\vec{r})$ from (18). We simply select $\overline{N}+1$ sampling values of the polar angle $\alpha$, regardless of $m$, say

$$\alpha = \alpha_p = p\,\frac{0.9\pi}{\overline{N}+1}, \qquad p = 1,\,2,\,\ldots,\,\overline{N}+1 \tag{19}$$

in (18). The $\overline{N}+1$ (18) can be written in the form of a matrix equation as follows:

$$[R^m]\left\{\Psi^m(\vec{r})\right\} = \left\{C^m(\vec{r})\right\} \tag{20}$$

with

$$\left\{C^m(\vec{r})\right\}_m = \frac{1}{N}\sum_{l=0}^{N-1} e^{jm(l2\pi/N)}e^{-j\vec{k}(\alpha_p,\,\beta_l)\cdot\vec{r}} \tag{21}$$

and

$$\left\{\Psi^m(\vec{r})\right\}_n = \Psi_{nm}(\vec{r}) = j_n(kr)P_n^m(\cos\theta)e^{jm\phi} \tag{22}$$

$$[R^m]_{p,\,n} = (-j)^n(2n+1)\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\alpha_p) \tag{23}$$

is a square $(\overline{N}+1 \times \overline{N}+1)$ matrix.

If we denote $[B^m] = [R^m]^{-1}$, the spherical wave function can be expressed as

$$\begin{aligned}\Psi_{nm}(\vec{r}) &= \sum_{p=1}^{\overline{N}+1} B^m(n,\,p)C^m(p)\\ &= \sum_{p=1}^{\overline{N}+1} B^m(n,\,p)\,\frac{1}{N}\sum_{l=0}^{N-1} e^{jm(l2\pi/N)}e^{-j\vec{k}(\alpha_p,\,\beta_l)\cdot\vec{r}}\\ &= \sum_{p=1}^{\overline{N}+1}\sum_{l=0}^{N-1} C_{pl}^{nm}e^{-j\vec{k}(\alpha_p,\,\beta_l)\cdot\vec{r}}\end{aligned} \tag{24}$$

TABLE I

COMPARISON OF THE NUMERICAL VALUES OF SPHERICAL WAVE FUNCTION: EXACT VALUE AND THE VALUE CALCULATED BY THE PLANE WAVE EXPANSION, WHERE $m = 2$, $n = 5$, $N_0 = 10$, $\phi = 30^0$ AND $r/\lambda = 0.5$

| $\theta$ (Degrees) | Re part of Exact | Im part of Exact | Re part of Expansion | Im part of Expansion |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 9 | 0.02436843 | 0.04220735 | 0.02436843 | 0.04220735 |
| 18 | 0.08143595 | 0.14105121 | 0.08143595 | 0.14105121 |
| 27 | 0.13278095 | 0.22998335 | 0.13278095 | 0.22998335 |
| 36 | 0.14093303 | 0.24410316 | 0.14093303 | 0.24410316 |
| 45 | 0.09250806 | 0.16022866 | 0.09250806 | 0.16022866 |
| 54 | 0.00734308 | 0.01271858 | 0.00734308 | 0.01271858 |
| 63 | -0.07198799 | -0.12468686 | -0.07198799 | -0.12468686 |
| 72 | -0.10436601 | -0.18076722 | -0.10436601 | -0.18076722 |
| 81 | -0.07399664 | -0.12816594 | -0.07399664 | -0.12816594 |
| 90 | 0 | 0 | 0 | 0 |

TABLE II

COMPARISON OF THE NUMERICAL VALUES OF SPHERICAL WAVE FUNCTION: EXACT VALUE AND THE VALUE CALCULATED BY THE PLANE WAVE EXPANSION, WHERE $m = 5$, $n = 8$, $N_0 = 10$, $\phi = 30^0$ AND $r/\lambda = 0.5$

| $\theta$ (Degrees) | Re part of Exact | Im part of Exact | Re part of Expansion | Im part of Expansion |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 9 | 0.00444399 | -0.00256574 | 0.00444326 | -0.00256532 |
| 18 | 0.11692227 | -0.0675051 | 0.11691046 | -0.06749828 |
| 27 | 0.6319977 | -0.36488404 | 0.63197446 | -0.36487062 |
| 36 | 1.59766578 | -0.92241277 | 1.59766805 | -0.92241408 |
| 45 | 2.32233101 | -1.34079844 | 2.32234561 | -1.34080686 |
| 54 | 1.83542817 | -1.05968495 | 1.83541579 | -1.0596778 |
| 63 | 0.09642698 | -0.05567214 | 0.09642546 | -0.05567127 |
| 72 | -1.55617505 | 0.89845809 | -1.55616251 | 0.89845084 |
| 81 | -1.59835033 | 0.92280799 | -1.59836176 | 0.92281459 |
| 90 | 0 | 0 | 0 | 0 |

where $C_{pl}^{nm} = (1/N)B_{np}^m e^{jml(2\pi/N)}$ and $B_{np}^m = B^m(n,\,p)$ is the $(n,\,p)$th element of the square matrix $[R^m]^{-1}$.

Equation (24) states that the spherical wave function $\Psi_{nm}(\vec{r})$ can be expanded by a series of *the same set* of plane wave functions, regardless of $n$, $m$ and vector $\vec{r}$, with $|\vec{r}| < R$. The information of the spectral harmonics is reflected in the amplitude of the plane wave expansion coefficient $C_{pl}^{nm}$. For a given pair of $m$ and $n$, there is the corresponding row and column, respectively, in the matrices $[R^m]$ and $[B^m]$. Therefore, for a given region with $|\vec{r}| < R$ and the corresponding set of spherical wave functions $\Psi_{nm}(\vec{r})$ that are needed to specify an arbitrary wave function in the region, i.e., $0 < n < \overline{N} = \{kR\}_I + N_0$, $-n \leq m \leq n$, we only need to *invert one matrix* to get the expansion coefficients for all the spherical wave functions of interest.

Having transformed the $(n,\,m)$th scalar spherical wave function into the plane wave series as given by (24), it is a straightforward task to deduce the associated $(n,\,m)$th multipole fields [2, pp. 414–416]

$$\vec{M}_{nm}(\vec{r}) = \nabla\Psi_{nm}(\vec{r}) \times \vec{r} \tag{25}$$

$$\vec{N}_{nm}(\vec{r}) = \frac{1}{k}\,\nabla \times \vec{M}_{nm}. \tag{26}$$

Using (24) and $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$ in (25) and (26) we can obtain the vector multipole fields in the Cartesian coordinates.

TABLE III
COMPARISON OF THE NUMERICAL VALUES OF SPHERICAL WAVE FUNCTION: EXACT VALUE AND THE VALUE CALCULATED BY THE PLANE WAVE EXPANSION, WHERE $m = 12$, $n = 14$, $N_0 = 40$, $\phi = 30^0$ AND $r/\lambda = 0.5$

| $\theta$ (Degrees) | Re part of Exact | Im part of Exact | Re part of Expansion | Im part of Expansion |
|---|---|---|---|---|
| 0 | 0 | 0 | -0.00001651 | 0.0000183 |
| 9 | 0.00002702 | 0 | 0.00002924 | 0.00001489 |
| 18 | 0.08815401 | 0 | 0.08815644 | -0.00000047 |
| 27 | 7.77629471 | 0 | 7.77629868 | -0.00001084 |
| 36 | 140.7471545 | 0 | 140.7471625 | -0.00002078 |
| 45 | 969.5378886 | 0 | 969.5378672 | -0.0000404 |
| 54 | 3249.984637 | 0 | 3249.98453 | -0.00003952 |
| 63 | 5673.310294 | 0 | 5673.310126 | -0.0000392 |
| 72 | 4290.343977 | 0 | 4290.343734 | -0.0000614 |
| 81 | -1451.477577 | 0 | -1451.477862 | -0.00005108 |
| 90 | -4964.03399 | 0 | -4964.034301 | -0.00005141 |

TABLE IV
COMPARISON OF THE NUMERICAL VALUES OF SPHERICAL WAVE FUNCTION: EXACT VALUE AND THE VALUE CALCULATED BY THE PLANE WAVE EXPANSION, WHERE $m = 5$, $n = 6$, $N_0 = 20$, $\phi = 30^0$ AND $r/\lambda = 1.0$

| $\theta$ (Degrees) | Re part of Exact | Im part of Exact | Re part of Expansion | Im part of Expansion |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 9 | 0.08959997 | -0.05173057 | 0.08959997 | -0.05173057 |
| 18 | 2.59503904 | -1.49824649 | 2.59503904 | -1.49824649 |
| 27 | 16.63950714 | -9.60682393 | 16.63950714 | -9.60682393 |
| 36 | 54.96380893 | -31.73336988 | 54.96380893 | -31.73336988 |
| 45 | 121.0419086 | -69.88357853 | 121.0419086 | -69.88357853 |
| 54 | 197.2571065 | -113.8864436 | 197.2571065 | -113.8864436 |
| 63 | 246.874899 | -142.5332894 | 246.874899 | -142.5332894 |
| 72 | 232.8304309 | -134.424712 | 232.8304309 | -134.424712 |
| 81 | 142.3828976 | -82.20480424 | 142.3828976 | -82.20480424 |
| 90 | 0 | 0 | 0 | 0 |

## III. NUMERICAL VERIFICATIONS

To validate the proposed transformation, numerical calculations for a variety of parameters are given here. Table I shows the comparison of the values of the spherical wave function from the original expression [(1)] and from the plane wave series expansion [(24)] for a small electrical size problem ($r/\lambda = 0.5$). The series expansion quickly converges to the eighth decimal place by setting $N_0 = 10$. When the orders of the spherical wave function are increased to $m = 5$ and $n = 8$, it can be seen from Table II that the relaxation integer constant $N_0$ provides only fifth decimal place accuracy. This phenomenon becomes more obvious in Table III, in which a high order case ($m = 12$, $n = 14$) is considered with the relaxation constant $N_0 = 40$.

To show the applicability of the proposed plane wave series expansion to the problem with large electrical size, the comparisons of the numerical values of the original expression and those of the plane wave series expansion are made for the cases of $r/\lambda = 1.0$, $r/\lambda = 2.0$ and $r/\lambda = 3.0$ in Table IV, Table V and Table VI, respectively. As can be perceived, the larger the electrical size, the greater the relaxation constant needed.

## IV. CONCLUSION

A finite plane wave series expansion for the scalar spherical wave functions has been presented in this paper. The formulation starts from the well-known spherical wave function expansion of a scalar plane wave and ends with a finite series expansion of plane wave functions for scalar spherical wave functions of the first kind. In the derivation of the series expansion, we have used the DFT and the fact that $j_n(kr) \approx 0$ when the order $n$ is sufficiently larger than the argument $kr$. For a given region for which $0 < r < R$, each of the spherical wave functions $\Psi_{nm}(r, \theta, \phi)$ can be represented by *the same set* of *plane wave functions* with different weighting coefficients for different ($n$, $m$). This will greatly facilitate the modal analysis of various practical guided wave and scattering problems involving 3-D irregularly shaped regions.

TABLE V
COMPARISON OF THE NUMERICAL VALUES OF SPHERICAL WAVE FUNCTION: EXACT VALUE AND THE VALUE CALCULATED BY THE PLANE WAVE EXPANSION, WHERE $m = 6$, $n = 9$, $N_0 = 30$, $\phi = 68^0$ AND $r/\lambda = 2.0$

| $\theta$ (Degrees) | Re part of Exact | Im part of Exact | Re part of Expansion | Im part of Expansion |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 9 | 3.52230852 | 3.91191992 | 3.52230852 | 3.91191992 |
| 18 | 183.6069842 | 203.9162144 | 183.6069842 | 203.9162144 |
| 27 | 1466.824967 | 1629.074166 | 1466.824967 | 1629.074166 |
| 36 | 4856.985287 | 5394.228644 | 4856.985287 | 5394.228644 |
| 45 | 8708.502419 | 9671.771771 | 8708.502419 | 9671.771771 |
| 54 | 8482.809101 | 9421.113949 | 8482.809101 | 9421.113949 |
| 63 | 2050.207705 | 2276.986336 | 2050.207705 | 2276.986336 |
| 72 | -5639.334794 | -6263.115798 | -5639.334794 | -6263.115798 |
| 81 | -6722.440842 | -7466.026929 | -6722.440842 | -7466.026929 |
| 90 | 0 | 0 | 0 | 0 |

TABLE VI
COMPARISON OF THE NUMERICAL VALUES OF SPHERICAL WAVE FUNCTION: EXACT VALUE AND THE VALUE CALCULATED BY THE PLANE WAVE EXPANSION, WHERE $m = 6$, $n = 9$, $N_0 = 40$, $\phi = 68^0$ AND $r/\lambda = 3.0$

| $\theta$ (Degrees) | Re part of Exact | Im part of Exact | Re part of Expansion | Im part of Expansion |
|---|---|---|---|---|
| 0 | 0 | 0 | 0.00000573 | 0.00000176 |
| 9 | 1.93168 | 2.14534798 | 1.93168353 | 2.14533685 |
| 18 | 100.6924684 | 111.8303155 | 100.6925661 | 111.8304248 |
| 27 | 804.4259714 | 893.4055511 | 804.4259956 | 893.4056404 |
| 36 | 2663.634172 | 2958.265447 | 2663.633991 | 2958.265485 |
| 45 | 4775.856475 | 5304.125971 | 4775.856542 | 5304.125909 |
| 54 | 4652.083312 | 5166.661947 | 4652.083439 | 5166.661926 |
| 63 | 1124.360685 | 1248.729048 | 1124.360623 | 1248.729096 |
| 72 | -3092.684861 | -3434.774511 | -3092.684856 | -3434.774519 |
| 81 | -3686.674365 | -4094.466688 | -3686.674365 | -4094.466699 |
| 90 | 0 | 0 | 0.00006139 | -0.00005115 |

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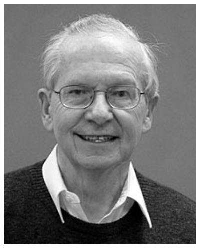

**Robert H. MacPhie** (S'57–M'63–SM'79–F'91–LF'97) received the B.A.Sc. degree in electrical engineering from the University of Toronto, Toronto, ON, Canada, in 1957 and the M.S. and Ph.D. degrees from the University of Illinois, Urbana, in 1959 and 1963, respectively.

In 1963, he joined the University of Waterloo, Waterloo, ON, Canada, as an Assistant Professor of electrical engineering. He is now an Adjunct Professor Emeritus of electrical and computer engineering. From 1991 to 1992, he was on sabbatical leave as a Professeur Associé at the University of Aix-Marseille I, France, in the Départment de Radioélectricité. He was a visiting Professor in the Department of Electronic Engineering, The Chinese University of Hong Kong in 2003. His current research interests focus on waveguide scattering theory, scattering from spherical bodies, linear antennas, and phased arrays.

**Ke-Li Wu** (M'90–SM'96) received the B.S and M.Eng degrees from Nanjing University of Science and Technology, China, in 1982 and 1985, respectively, and the Ph.D. degree from Laval University, Laval, QC, Canada, in 1989.

From 1989 to 1993, he was with the Communications Research Laboratory, McMaster Univeristy, Hamilton, ON, Canada, as a Research Engineer and a Research Group Manager. He joined the Corporate R&D Division, Com Dev International, Canada in March 1993, where he had been a Principal Member of Technical Staff in charge of developing advanced electromagnetic design software for various microwave subsystem for communication satellite and wireless communications. He has been with the Department of Electronic Engineering, The Chinese University of Hong Kong, since October 1999, where he is currently a Professor. He has published widely in the areas of EM modeling and microwave and antenna engineering and holds one Canadian patent and one U.S. patent. He contributed to *Finite Element and Finite Difference Methods in Electromagnetics Scattering* (Elsevier: Amsterdam, The Netherlands, 1990) and to *Computational Electromagnetics* (Elsevier: Amsterdam, The Netherlands, 1991). His current research interests include all the aspects related to low temperature co-fired ceramic (LTCC) multichip modules (MCM) from EM modeling, and design methodology to various applications.

Dr. Wu was a recipient of the URSI Young Scientist Award in 1992. He was awarded the Industry Feedback Award from Telecommunication Research Institute of Ontario in 1993 and Com Dev's Achievement Award in 1998.