Detecting Fixed Amplitude Signals in a Fuzzy Gaussian Environment

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ABSTRACT

In practical signal detection scenarios, parameters of a random process are often uncertain. In this paper, we model such uncertainties as fuzzy parameters of a stationary random process. A fuzzy Neyman-Pearson hypothesis test concept which accepts any number of fuzzy parameters is presented. A suitable decision rule is developed by applying theory for ordering fuzzy numbers, and stated in terms of a *fuzzy threshold*. A *defuzzifying threshold* is then applied to produce a crisp decision rule. The concepts developed here are applied to Neyman-Pearson detection of a fuzzy fixed amplitude signal in Gaussian noise with fuzzy variance.

1. INTRODUCTION

The majority of real signal detection problems relate to phenomena which cannot be measured precisely or are otherwise uncertain. Traditionally, these phenomena are modeled as random processes with parameters that are crisp-valued, interval-valued, or random. Decision rules are generally obtained using hypothesis testing methods [1, 2]. However, a number of approaches to decision theory using fuzzy logic [3] which are applicable to signal detection have also been proposed [4,5].

Saade and Schwarzlander developed [4] a fuzzy extension to binary hypothesis testing which handles the superposition of randomness and fuzziness. In their work, a fuzzy likelihood ratio test is presented for which fuzziness in a single parameter under one hypothesis only is modeled. As an example, they consider detection of a fixed amplitude fuzzy signal in **crisp** Gaussian noise. In this paper, we expand the model to allow for an arbitrary number of fuzzy parameters under each hypothesis and apply the model to detect a fuzzy signal in **fuzzy** Gaussian noise.

Likelihood ratio tests implement decision rules by evaluating and then thresholding the likelihood ratio for a number of observations [1, 2]. In the fuzzy case, the likelihood ratio is a function of fuzzy parameters and so is in general a fuzzy number. Therefore, the decision rule is also fuzzy. By adapting fuzzy ordering methods [6], we derive a fuzzy threshold, induced by a conventional crisp threshold, which quantifies the degree of certainty in the detection decision.

2. THE FUZZY NEYMAN-PEARSON TEST

In this section, we illustrate a method for Neyman-Pearson hypothesis testing [1] which models fuzziness in parameters of the hypotheses being tested. We first summarize the Neyman-Pearson test for crisp parameters and then expand the theory to account for fuzzy parameters.

The conventional, crisp Neyman-Pearson test operates on a class of disjoint sets and seeks to establish which set contains the correct value of an unknown parameter. The contribution of the Neyman-Pearson test is to minimize the probability of Type 2 error for constrained probability of Type 1 error [2]. In many applications, such as binary signal detection [7], only two hypotheses are considered. From here on, we focus our attention to testing simple hypotheses [1] only.

The crisp Neyman-Pearson test can now be formulated as follows. The two hypotheses under consideration are

$$\begin{aligned} H_0 &: A = A_0 \\ H_1 &: A = A_1 \end{aligned}, \tag{1}$$

where A is an unknown parameter of a random variable X. X may also depend on other parameters, $B_1, \ldots, B_k$, which are known, and has probability density function denoted by $f_X(x; A, B_1, \ldots, B_k)$. The likelihood ratio is then defined as

$$\Lambda_X(x; A_0, A_1, B_1, \ldots, B_k) := \frac{f_X(x; A_1, B_1, \ldots, B_k)}{f_X(x; A_0, B_1, \ldots, B_k)}, \tag{2}$$

where x is an observation of random variable X. The Neyman-Pearson decision rule is

$$\Lambda_X(x; A_0, A_1, B_1, \ldots, B_k) \underset{H_0}{\overset{H_1}{\gtrless}} g \;, \tag{3}$$

where g is a constant threshold determined by the chosen probability of Type 1 error, which may be found by the method of Lagrange multipliers [2].

We now extend the definition of the likelihood ratio (2) to a function of fuzzy numbers. Henceforth, we will denote fuzziness by the symbol '~', and write the membership function [3] of a fuzzy set $\tilde{X}$ as $m_{\tilde{X}}(x)$. Applying the Extension Principle for fuzzy sets [8] to (2), we obtain the fuzzy likelihood ratio, $\tilde{\Lambda}_{\tilde{X}}(x)$. This is a fuzzy number with membership function

$$m_{\tilde{\Lambda}_X(x)}(z) = \sup_{z = \Lambda_X(x; a_0, a_1, b_1, \ldots, b_k)} \left\{ \min \left[\begin{array}{l} m_{\tilde{A}_0}(a_0), m_{\tilde{A}_1}(a_1), \\ \quad m_{\tilde{B}_1}(b_1), \ldots, m_{\tilde{B}_k}(b_k) \end{array} \right] \right\}, \tag{4}$$

which generalizes the expression given in [4]. The decision rule (3) can therefore be written in terms of the fuzzy likelihood ratio as

$$\tilde{\Lambda}_X(x) \underset{H_0}{\overset{H_1}{\gtrless}} g \,. \tag{5}$$

To implement decision rule (5) we must order the fuzzy likelihood ratio, $\tilde{\Lambda}_X(x)$, and the crisp threshold, g. Figure 1 shows a plot of the fuzzy likelihood ratio for three typical values of X, together with the crisp threshold.

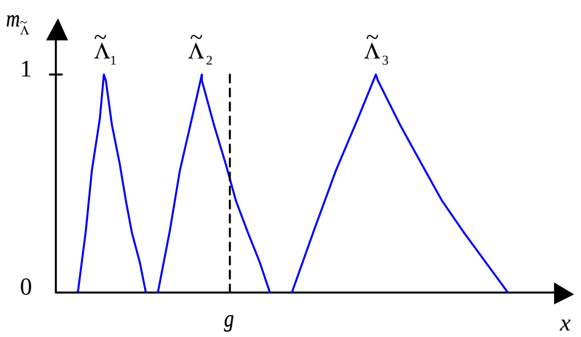

Figure 1. Plot of Fuzzy Likelihood Ratio and Crisp Threshold.

It is clear from Figure 1 that $\tilde{\Lambda}_1$ is smaller than the crisp threshold while $\tilde{\Lambda}_3$ is larger than the crisp threshold. In these two cases, therefore, decision rule (5) results in the crisp decisions H_0 and H_1, respectively. However, it is not clear what decision should be made for $\tilde{\Lambda}_2$ since the order of $\tilde{\Lambda}_2$ and g is ambiguous. In [6], such ambiguity is handled by ordering them crisply according to the *total distance criterion*, in effect defuzzifying the decision rule. In this paper, however, we obtain a fuzzy ordering for $\tilde{\Lambda}_2$ and g by defining an ordering membership function which assigns membership values to each fuzzy event, "$\tilde{\Lambda}_2$ is greater than g" and "g is greater than $\tilde{\Lambda}_2$". Specifically, we select a fuzzy ordering membership function, based on the total distance criterion, defined by

$$\Omega\left(\tilde{Y}_1,\tilde{Y}_2\right) := 1-\frac{d\left(\tilde{Y}_1,\tilde{Y}_1\wedge\tilde{Y}_2\right)}{d\left(\tilde{Y}_1,\tilde{Y}_1\wedge\tilde{Y}_2\right)+d\left(\tilde{Y}_2,\tilde{Y}_1\wedge\tilde{Y}_2\right)}\,, \tag{6}$$

where $d\left(\tilde{Y}_1,\tilde{Y}_2\right)$ is the total distance between fuzzy numbers $\tilde{Y}_1$ and $\tilde{Y}_2$ [6].

When applied to decision rule (5), we then obtain the fuzzy Neyman-Pearson test for binary hypotheses under uncertainty:

$$\begin{aligned} &\text{Accept } H_0 \text{ with degree } \Omega\left(g,\tilde{\Lambda}_X(x)\right) \\ &\text{Accept } H_1 \text{ with degree } \Omega\left(\tilde{\Lambda}_X(x),g\right) \end{aligned}\,. \tag{7}$$

In general, each hypothesis has non-crisp membership reflecting the uncertainty in the decision due to **fuzziness**. Uncertainty due to **randomness** has already been accounted for by the choice of crisp threshold, g.

Expressed as a function of x, we refer to

$$\Omega_X(x) := \Omega\left(\tilde{\Lambda}_X(x),g\right) \tag{8}$$

as the *fuzzy threshold* of the random variable X, to contrast it with the crisp threshold, g. The fuzzy threshold is an increasing function of x as shown in Figure 2, similar in form to a cumulative distribution function. Indeed, previously a cumulative distribution function has itself been used as a kind of fuzzy threshold [9].

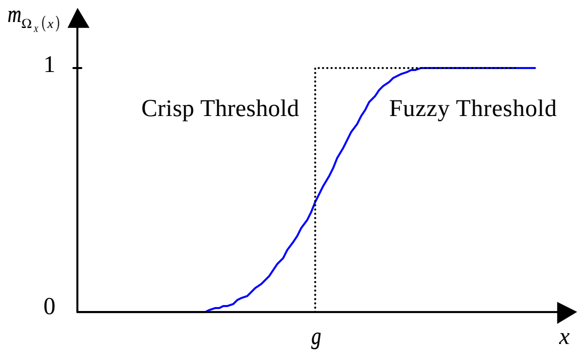

Figure 2. Comparison of Crisp and Fuzzy Threshold.

If further processing is required, such as in binary integration [7] or data fusion [10], the output of the fuzzy threshold may be processed without "defuzzification". In the case of binary integration, the detection performance resulting from the use of a fuzzy threshold is markedly greater than for its traditional crisp counterpart [10]. However, in the absence of further processing, a crisp decision rule can be obtained by applying a *defuzzifying threshold*:

$$\Omega_X(x) \underset{H_0}{\overset{H_1}{\gtrless}} w \,. \tag{9}$$

In order to determine the performance of the fuzzy Neyman-Pearson test, we must calculate probabilities associated with each hypothesis. The probability of Type 1 error, or *false alarm rate*, of decision rule (9) for crisp parameters is given by

$$\begin{aligned} P^{FA} &:= \Pr\left\{\Omega_X(x)>w \mid H_0\right\}, \\ &= \Pr\left\{x>\Omega_X^{-1}(w) \mid H_0\right\}, \\ &= 1-F_{X|H_0}\left(\Omega_X^{-1}(w)\right), \end{aligned} \tag{10}$$

where $F_{X|H_0}(z)$ is the distribution function of X under H_0. This result is extended to account for fuzzy parameters by again applying the Extension Principle for fuzzy numbers. The false alarm rate is then given by the fuzzy set $\tilde{P}^{FA}$ with membership

$$m_{\tilde{P}^{FA}}(z)=\sup_{z\,=\,1-F_{X|H_0}\left(\Omega_X^{-1}(w)\right)}\left\{\min\left[\begin{matrix} m_{\tilde{A}_0}(a_0), m_{\tilde{A}_1}(a_1), \\ m_{\tilde{B}_1}(b_1),\ldots,m_{\tilde{B}_k}(b_k)\end{matrix}\right]\right\}. \tag{11}$$

Similarly, the probability of correctly selecting hypothesis H_1, the *detection rate*, is given by the fuzzy set $\tilde{P}^{D}$ with membership

$$m_{\tilde{P}^D}(z) = \sup_{z = 1 - F_{X|H_1}(\Omega_X^{-1}(w))} \left\{ \min \left[\begin{array}{l} m_{\tilde{A}_0}(a_0), m_{\tilde{A}_1}(a_1), \\ m_{\tilde{B}_1}(b_1), \ldots, m_{\tilde{B}_k}(b_k) \end{array} \right] \right\}. \quad (12)$$

3. FUZZY NEYMAN-PEARSON DETECTION IN GAUSSIAN NOISE

We now consider the performance of the fuzzy Neyman-Pearson test applied to a simple signal detection problem – detecting a fixed amplitude signal in Gaussian noise. We first develop the standard likelihood ratio for crisp signal amplitude and noise variance. We go on to derive the fuzzy likelihood ratio and detection performance for fuzzy signal and noise.

Consider a Gaussian channel which is sampled to provide n independent observations of a random variable. From this sample we wish to determine the presence a particular signal with amplitude A. We state this problem as the hypothesis test

$$\begin{array}{l} H_0: \text{"Signal Not Present"} \\ H_1: \text{"Signal Present"} \end{array} . \quad (13)$$

If the signal is present, observations will be Gaussian random variables with mean A. Otherwise, observations will be zero-mean Gaussian random variables. In each case, the noise variance, S^2, is assumed known. This corresponds to hypothesis test (1) with $A_0 = 0$, $A_1 = A$, and a single known parameter, $B_1 = S^2$. The hypothesis test may therefore be restated as

$$\begin{array}{l} H_0:\ X \sim N(0, S^2) \\ H_1:\ X \sim N(A, S^2) \end{array}, \quad (14)$$

where $X \sim N(A, S^2)$ denotes that X is a Gaussian random variable with mean A and variance S^2.

The likelihood ratio is given in terms of the observation mean, $\overline{x}$, by [11]

$$\Lambda_{\overline{X}}(\overline{x}; A, S^2) = \exp\left[-\frac{n}{2}\frac{A}{S^2}(A - 2\overline{x}) \right]. \quad (15)$$

We now turn to the case where the signal amplitude and noise variance are both fuzzy. We again denote fuzziness by the symbol "~". Combining equations (4) and (15), we obtain the fuzzy likelihood ratio, $\tilde{\Lambda}_{\overline{X}}(\overline{x})$, in terms of its membership function

$$m_{\tilde{\Lambda}_{\overline{X}}(\overline{x})}(z) = \sup_{z = \exp\left[-\frac{n}{2}\frac{a}{s^2}(a - 2\overline{x}) \right]} \left\{ \min \left[m_{\tilde{A}}(a), m_{\tilde{S}^2}(s^2) \right] \right\}. \quad (15)$$

In the results which follow, we use the notation $\tilde{Y} = \langle C, w \rangle$ to denote the triangular fuzzy number [8] with membership function

$$m_{\tilde{Y}}(x) = 1 - \frac{|x - C|}{w} \quad : \ |x - C| \le w . \quad (16)$$

Figure 3 shows the fuzzy likelihood ratio for various values of $\overline{x}$ when $\tilde{A}$ and $\tilde{S}^2$ are both given by the fuzzy number <1, 0.3>. The likelihood ratio is particularly sensitive to fuzziness in parameters $\tilde{A}$ and $\tilde{S}^2$ as $\overline{x}$ increases. Since the likelihood ratio is an exponential function of the observation mean, $\overline{x}$, uncertainties in the parameters are amplified for larger $\overline{x}$.

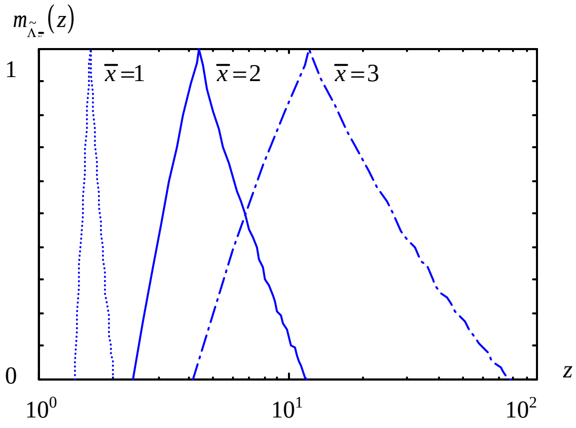

Figure 3. Plot of Fuzzy Likelihood Ratio for Various $\overline{x}$.

The resultant fuzzy threshold is shown in Figure 4 for several values of the fuzzy variance and fixed signal amplitude, $\tilde{A}$ = <1, 0.3>. When fuzziness in the noise variance is relatively small, the fuzzy threshold is steep producing a only small region in which the decision is fuzzy. However, as the fuzziness in the noise variance increases the fuzzy decision region expands. Thus increased fuzziness in the system parameters gives rise to increased uncertainty in the decision.

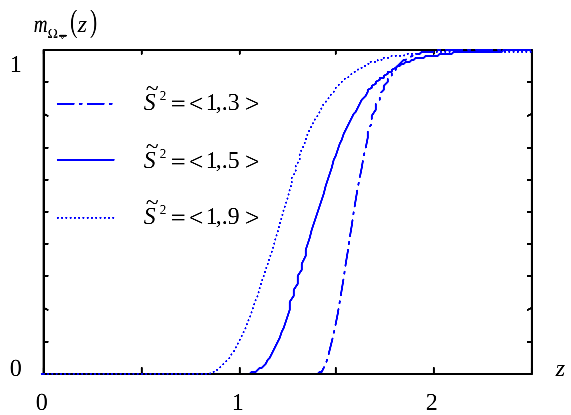

Figure 4. Plot of Fuzzy Threshold for Various $\tilde{S}^2$.

The performance of the detector can be obtained by applying equations (12) and (13) in Section 2. Figure 5(a) shows a plot of the fuzzy false alarm rate for several values of the defuzzifying threshold, w. Figure 5(b) shows the corresponding fuzzy detection rate. The figures show clearly that both the false alarm rate and detection rate increase as the defuzzifying threshold is decreased, similar to the behavior of conventional detectors when the threshold is decreased.

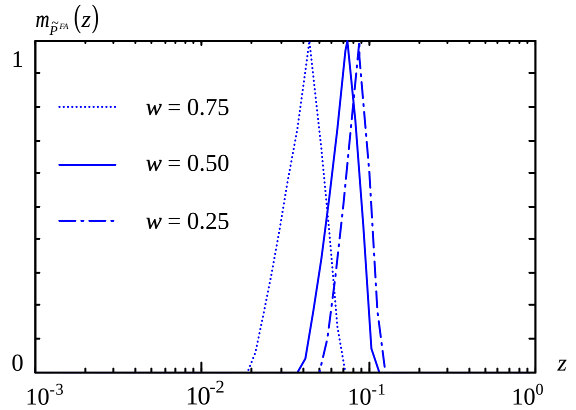

Figure 5(a). Plot of Fuzzy False Alarm Rate for Various w.
$\tilde{A} = <1, 0.3>$, $\tilde{S}^2 = <1, 0.3>$, $g = 3$.

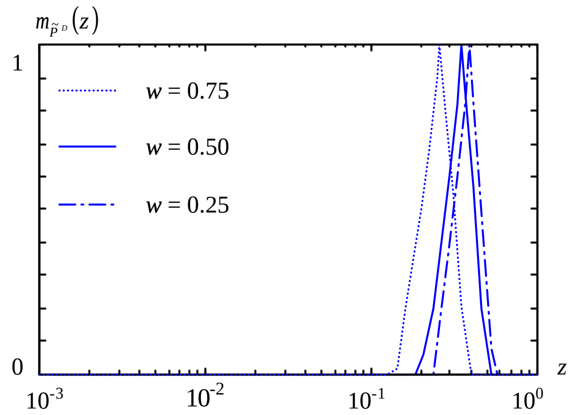

Figure 5(b). Plot of Fuzzy Detection Rate for Various w.
$\tilde{A} = <1, 0.3>$, $\tilde{S}^2 = <1, 0.3>$, $g = 3$.

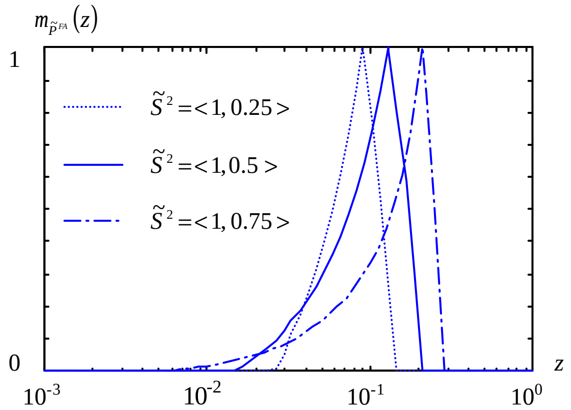

Figure 6(a). Plot of Fuzzy False Alarm Rate for Various $\tilde{S}^2$
$\tilde{A} = <1, 0.3>$, $g = 3$, $w = 0.5$.

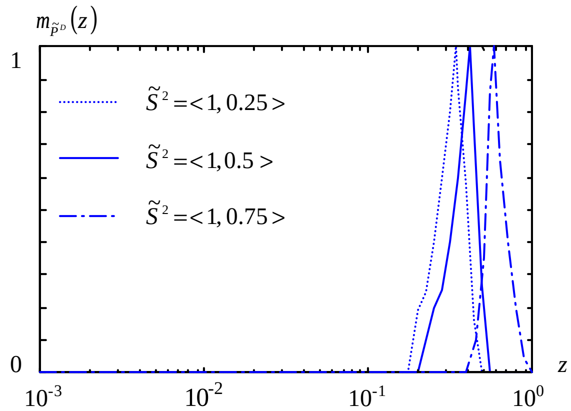

Figure 6(b). Plot of Fuzzy Detection Rate for Various $\tilde{S}^2$
($\tilde{A} = <1, 0.3>$, $g = 3$, and $w = 0.5$).

Figures 6(a) and 6(b) show the detector's performance in response to increasing fuzziness in the noise variance for $\tilde{A} = <1, 0.3>$, $g = 3$, and $w = 0.5$. The false alarm rate is sensitive to fuzziness in the noise variance. However, the detection rate is relatively insensitive to such uncertainty.

4. CONCLUSION

A method for performing Neyman-Pearson binary hypothesis testing for fuzzy parameters has been presented. A fuzzy threshold is derived by comparing the order of the likelihood ratio, which is fuzzy, and the crisp threshold. The output of this threshold describes a decision rule which assigns membership values to each hypothesis. In general, the decision rule is fuzzy however a crisp decision rule can be obtained by applying a defuzzifying threshold.

In Section 3, the fuzzy Neyman-Pearson test has been applied to detection of a fuzzy fixed amplitude signal in Gaussian noise with fuzzy variance. By deriving the fuzzy false alarm rate and detection rate, we may determine the impact of fuzziness in the system parameters on the detector's performance. The false alarm rate is found to be sensitive to fuzziness in the noise variance, but the detection rate is not. In this way, we are able to assess the accuracy of general Neyman-Pearson hypothesis tests under fuzzy uncertainty.

5. REFERENCES

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