ERG 2012B
Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #9
Taylor Series and Laurent Series

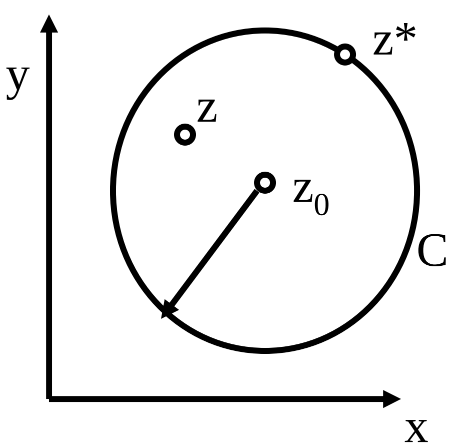
Taylor Series

Every analytic function f(z) can be represented by a power series which is called **Taylor series of f(z)**

Taylor's formula

We start with Cauchy's integral formula

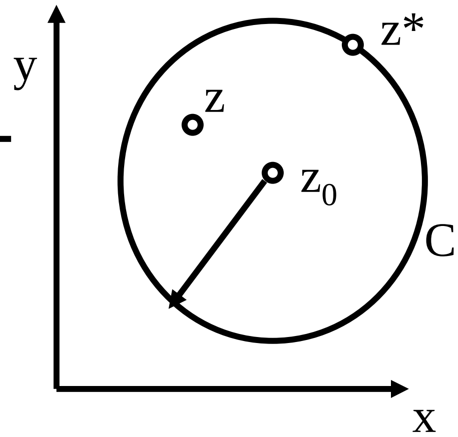
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^*-z}\, dz^*$$



where z lies inside C. Take C to be a circle of radius r, center z_0, then z* is on C

Taylor Series

Next we can write:



$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0-(z-z_0)} = \frac{1}{(z^*-z_0)(1-(z-z_0)/(z^*-z_0))}$$

Since z* is on C while z is inside C we have:

$$|(z-z_0)/(z^*-z_0)| < 1$$

And remembering that

$$\frac{1}{1-q} = 1 + q + \ldots\ldots + q^n + \frac{q^{n+1}}{1-q}$$ & letting $q = (z-z_0)/(z^*-z_0)$

Hence: $$\frac{1}{z^*-z} = \frac{1}{z^*-z_0}\left[1 + \frac{z-z_0}{z^*-z_0} + \left(\frac{z-z_0}{z^*-z_0}\right)^2 + \ldots. + \left(\frac{z-z_0}{z^*-z_0}\right)^n\right] + \frac{1}{z^*-z}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}$$

(continued on next slide)

Taylor Series

Hence:
$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0}\left[1 + \frac{z-z_0}{z^*-z_0} + \left(\frac{z-z_0}{z^*-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{z^*-z_0}\right)^n\right] + \frac{1}{z^*-z}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}$$

Substituting this into Cauchy's Integral formula we get:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z_0}\,dz^* + \frac{z-z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}\,dz^* + \dots\dots + \frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}\,dz^* + R_n(z)$$

Where $R_n(z)$ is given by:

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}\,dz^*$$

(continued on next slide)

Taylor Series

Using the integral formula for derivatives of analytic functions

$$f(z)=f(z_0) + \frac{z-z_0}{1!} f^{/}(z_0) + \frac{(z-z_0)^2}{2!} f^{//}(z_0) + ... + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)+R_n(z_0)$$

This is known as **Taylor's Formula** and $R_n(z)$ is called the **remainder**

If we let n approach infinity, we obtain:

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!} (z-z_0)^m$$

This is called the **Taylor Series** of f(z) with center z_0

The particular case when $z_0=0$ is called the **Maclaurin series**

Taylor Series

The Taylor series will converge and represent f(z) iff

$$\lim_{n\to\infty} R_n(z) = 0$$

Proof: from the definition of $R_n(z)$

$$\left|R_n(z)\right| = \frac{|z-z_0|^{n+1}}{2\pi}\left|\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}dz^*\right|$$

Since z* is on C, z is inside C we have:

$$|z^*-z_0| = r,\ \ |z^*-z| > 0,\ \ |z-z_0|/r < 1$$

Since f(z) is analytic inside and on C, $|f(z)/(z^*-z)| \leq M$ (bounded)

therefore by the ML inequality

$$|R_n(z)| \leq [|z-z_0|^{n+1}/2\pi]M2\pi r/r^{n+1} = Mr|(z-z_0)/r|^{n+1} \to 0 \text{ as } n\to\infty$$

Taylor Theorem

Taylor's Theorem summarizes the preceding:
Let $f(z)$ be analytic in a domain D and let $z=z_0$ be any point in D. Then there exists precisely one power series with center z_0 that represents $f(z)$. This series is of the form:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{where } a_n = (1/n!)\, f^{(n)}(z_0)$$

This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic. The remainders $R_n(z)$ can be represented as before.

The coefficients satisfy: $|a_n| \leq M/r^n$
where M is the maximum of $|f(z)|$ on the circle $|z-z_0|=r$

Taylor Theorem

The inequality $|a_n| \leq M/r^n$ follows from Cauchy's inequality earlier.

The formula for derivatives of analytic functions gives the coefficients

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

integrated ccw around a simple closed path containing z_0

Singular Points

Singular points of an analytic function f(z) are points at which f(z) ceases to be analytic.

If f(z) is not differentiable at the $z=c$, but every disk with center c contains points at which f(z) is differentiable then that point is called a **singular point of f(z)**

We say that f(z) **has a singularity** at $z=c$.

E.g $1/(1-z)$ at $z = 1$

$\tan z$ at $\pm\pi/2, \pm3\pi/2......$

Important Special Taylor Series

Geometric Series

Let $f(z) = 1/(1-z)$. Then we have $f^{(n)}(z)=n!/(1-z)^{n+1}$, $f^{(n)}(0)=n!$

Taylor's Theorem:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{where } a_n = (1/n!)\, f^{(n)}(z_0)$$

For $z_0=0$ $\quad a_n = (1/n!)n! = 1$

So the Maclaurin expansion of $1/(1-z)$ is the geometric series

$$1/(1-z) = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots\ldots \qquad (|z| < 1)$$

$f(z)$ is singular at $z=1$; which lies on the circle of convergence.

Important Special Taylor Series II

Exponential Function

e^z is analytic for all z and $(e^z)' = e^z$.

Taylor's Theorem:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{where } a_n = (1/n!)\, f^{(n)}(z_0)$$

For $z_0=0$ $a_n = (1/n!)$

So the Maclaurin expansion of e^z is the geometric series

$$e^z = \sum_{n=0}^{\infty} z^n/n! = 1 + z + z^2/2! + \ldots\ldots$$

If we let z = *i*y and separate the series into real and imaginary

$$e^{iy} = \sum_{n=0}^{\infty} (iy)^n/n! = \sum_{n=0}^{\infty} (-1)^k y^{2k}/(2k)! + i\sum_{n=0}^{\infty} (-1)^k y^{2k+1}/(2k+1)!$$

The two series are simply the series for sin and cos and we rediscover the Euler formula $e^{iy} = \cos y + i \sin y$

Important Special Taylor Series III

Trigonometric & Hyperbolic Functions

By substituting series for e^z in formula for cos and sin

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)! = 1 - z^2/2 + z^4/4! - \ldots\ldots$$

$$\sin z = \frac{1}{2}(e^{iz} - e^{-iz})$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)! = z - z^3/3! + z^5/5! - \ldots\ldots$$

Similarly for the hyperbolic functions:

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

$$\cosh z = \sum_{n=0}^{\infty} z^{2n}/(2n)! = 1 + z^2/2 + z^4/4! + \ldots\ldots$$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$\sinh z = \sum_{n=0}^{\infty} z^{2n+1}/(2n+1)! = z + z^3/3! + z^5/5! + \ldots\ldots$$

Important Special Taylor Series IV

Logarithm

From Taylor's Theorem for $z_0=0$:

$$f(z) = \sum_{n=0}^{\infty} a_n(z)^n \quad \text{where } a_n = (1/n!)\, f^{(n)}(0)$$

$$\mathrm{Ln}(1+z) = \sum_{n=0}^{\infty} (-1)^n z^{n+1}/(n+1) = z - z^2/2 + z^3/3 - \ldots\ldots \qquad (|z|<1)$$

Replacing z by -z and multiply both sides by -1

$$\mathrm{Ln}(1/(1-z)) = \sum_{n=0}^{\infty} z^{n+1}/(n+1) = z + z^2/2 + z^3/3 + \ldots\ldots \qquad (|z|<1)$$

adding both series:

$$\mathrm{Ln}((1+z)/(1-z)) = \sum_{n=0}^{\infty} 2z^{2n+1}/(2n+1) = 2(z + z^3/3 + z^5/5 + \ldots\ldots)$$

$$(|z|<1)$$

Theorem 2

Every power series with a nonzero radius of convergence is the Taylor series of the function represented by that series *or to put it another way* is the Taylor series of its sum

Proof: Consider any power series with positive radius of convergence R and call its sum f(z); thus

$$f(z) = a_0+a_1(z-z_0)+a_2(z-z_0)^2+....$$

And $f^{/}(z) = a_1 + 2a_2(z-z_0)+.......$

More generally

$$f^{(n)}(z) = n!a_n+(n+1)n.....3\times 2\times 1\ a_{n+1}(z-z_0)+.......$$

if we set $z=z_0$ we obtain:

$$f(z_0) = a_0,\ \ f^{/}(z_0) = a_1,.......\ f^{(n)}(z_0) = n!a_n$$

This is identical to the terms in the Taylor Theorem.......

Finding Taylor Series of Functions

Example 1 Find the Maclaurin series of $f(z)=1/(1+z^2)$

Solution: by substitution into $1/(1-z) = \Sigma z^n$

$$1/(1+z^2) = \sum_{n=0}^{\infty} 1/(1-(-z^2)) = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

$$= 1 - z^2 + z^4 - z^6 + \ldots\ldots\ldots\ldots\ldots \qquad |z|<1$$

Example 2 Find the Maclaurin series of $f(z) = \tan^{-1}z$

Solution: by integration of previous example term by term

$$f'(z) = \quad 1/(1+z^2) = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

Integrating term by term and using $f(0)=0$

$$\Rightarrow \tan^{-1}z = \sum_{n=0}^{\infty} [(-1)^n/(2n+1)]\, z^{2n+1} = z - z^3/3 + z^5/5 - \ldots\ldots\ldots \; |z|<1$$

Example 3

Develop $1/(c-bz)$ in powers of $z-a$ where $c-ab \neq 0$ and $b \neq 0$

Solution:

$$1/(c-bz) = 1/(c-ab-b(z-a)) = 1/[(c-ab)(1-b(z-a)/(c-ab))]$$

$$= 1/(c-ab)\sum_{n=0}^{\infty}[b(z-a)/(c-ab)]^n = \sum_{n=0}^{\infty}(b^n/(c-ab)^{n+1})(z-a)^n$$

$$= 1/(c-ab) + b(z-a)/(c-ab)^2 + b^2(z-a)^2/(c-ab)^3 + \ldots.$$

which converges for

$|b(z-a)/(c-ab)| < 1$, i.e. $|z-a| < |(c-ab)/b| = |(c/b)-a|$

Example 4

Find the Taylor series of f(z) with center $z_0=1$, where

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

Solution:

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{(z-3)} = \frac{1}{[3+(z-1)]^2} - \frac{2}{2-(z-1)}$$

$$= \frac{1}{9}\frac{1}{[1+(z-1)/3]^2} - \frac{1}{1-(z-1)/2}$$

expressing f(z) as a sum of partial fractions

Note binomial series $1/(1+z)^m = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$

$$= 1-mz+(-m(-m-1)z^2/2! + -m(-m-1)(-m-2)z^3/3!+\ldots$$

so that $f(z) = 1/9 \sum_{n=0}^{\infty} \binom{-2}{n} ((z-1)/3)^n - \sum_{n=0}^{\infty} ((z-1)/2)^n$

since z=3 is nearest singularity to z=1 series converges for |z-1|<2

since $\binom{-2}{n} = \frac{(-2)(-3)....(-(n+1))}{1.2.3.........n} = (-1)^n (n+1)$

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{(-1)^n(n+1)}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n = -\frac{8}{9} - \frac{31}{54}(z-1) - \frac{23}{108}(z-1)^2$$

Example 5

Find the Maclaurin series $f(z) = \tan z$

Solution:

$f'(z) = \sec^2 z = 1+\tan^2 z = 1 + f^2(z)$; $f(0)=0$, $f'(0)=1$

$f'' = 2ff'$, $f''(0)=0$

$f''' = 2(f')^2 + 2ff''$, $f'''(0)=2$, $f'''(0)/3! = 1/3$

$f^{(4)} = 6f'f'' + 2ff'''$, $f^{(4)}(0)=0$

$f^{(5)} = 6(f'')^2 + 8f'f''' + 2ff^{(4)}$, $f^{(5)}(0)=16$, $f^{(5)}(0)/5!=2/15$

$\tan z = z + z^3/3 + 2z^5/15 + 17z^7/315 + \ldots\ldots..$ $(|z|<\pi/2)$

Example 6

Find the Maclaurin series of tan z by using those of cos & sin

Solution:

since tan z is odd, the desired expansion will be of the form

$$\tan z = a_1 z + a_3 z^3 + a_5 z^5 + \dots\dots\dots$$

Using sin z = tan z cos z

$$z - z^3/3! + z^5/5! - \dots = (a_1 z + a_3 z^3 + a_5 z^5 + ..)(1 - z^2/2! + z^4/4! - ..)$$

implies $1 = a_1$, $-1/3! = -a_1/2! + a_3$, $1/5! = a_1/4! - a_3/2! + a_5$,.....

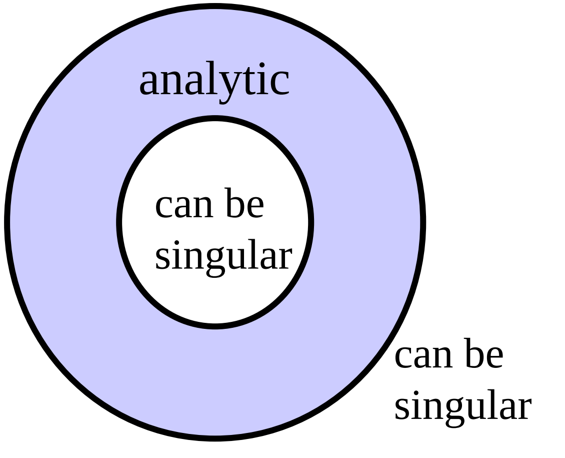
therefore $a_1 = 1$, $a_3 = 1/3$, $a_5 = 2/15$,......

Laurent Series

In applications you often need to expand a function around a point at which it is no longer analytic, but is singular.

Taylor’s Theorem no longer applies.

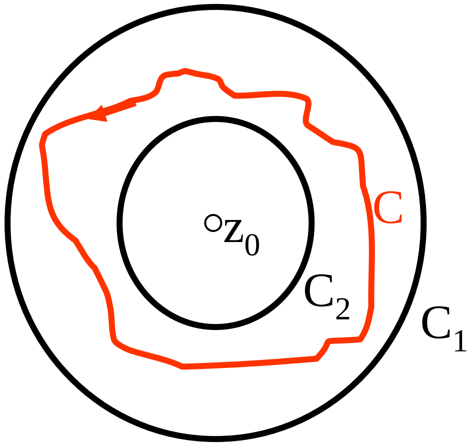
We need a new type of series – Laurent Series – which is convergent in an annulus in which f(z) is analytic and outside of which f(z) may have singular points



Laurent Series

Laurent's Theorem

If f(z) is analytic on two concentric circles C_1 and C_2 with center z_0 and in the annulus between them, then f(z) can be represented by the **Laurent series**



$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n/(z-z_0)^n$$

$$= a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots$$

$$\ldots.+b_1/(z-z_0) + b_2/(z-z_0)^2+\ldots.$$

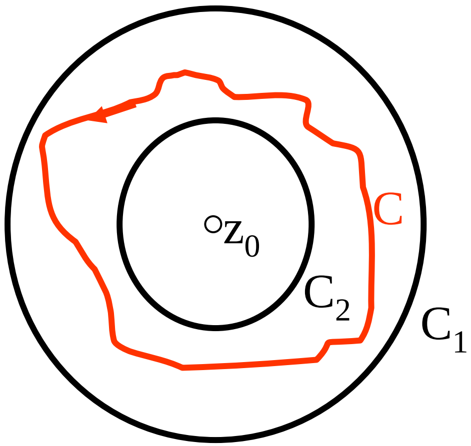
The coefficients of this Laurent series are given by the integrals

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}\, dz^*, \qquad b_n = \frac{1}{2\pi i}\oint_C (z^*-z_0)^{n-1} f(z^*)\,dz^*$$

each integral being taken ccw around any simple closed path C that lies in the annulus and encircles the inner circle.

Laurent Series

The series converges and represents f(z) in the open annulus obtained from the given annulus by continuously increasing the circle C_1 and decreasing C_2 until each of the two circles reaches a point were f(z) is singular.



In the important special case that z_0 is the only singular point of f(z) inside C_2 this circle can be shrunk to the point z_0, giving convergence in a disk except at the center.

The Laurent series can also be written (replacing b_n by a_{-n}):

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n , \qquad a_n = \frac{1}{2\pi i}\int_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}\, dz^*$$

Laurent Series

Proof: From Cauchy's integral Formula we have:

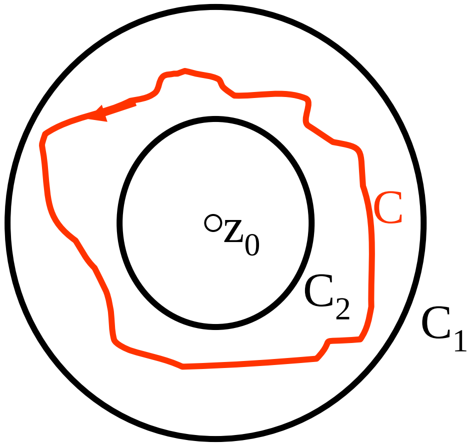
$$f(z) = \frac{1}{2\pi i}\oint_{C1} \frac{f(z^*)}{(z^*-z)}\,dz^* - \frac{1}{2\pi i}\oint_{C2} \frac{f(z^*)}{(z^*-z)}\,dz^*$$

1st integral like Taylor Theorem

$$\frac{1}{2\pi i}\oint_{C1} \frac{f(z^*)}{(z^*-z)}\,dz^* = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

with coefficients

$$a_n = \frac{1}{2\pi i}\oint_{C1} \frac{f(z^*)}{(z^*-z_0)^{n+1}}\,dz^*$$



C_1 can be replaced by C by the principle of deformation of path

Laurent Series

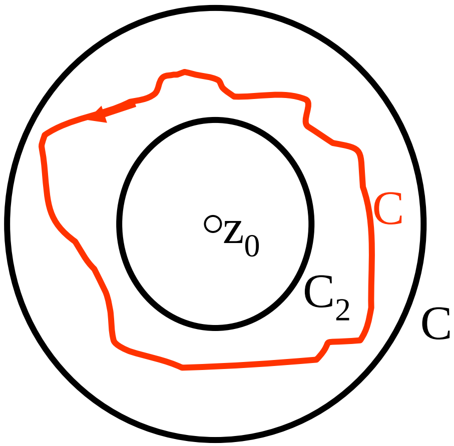
Proof: From Cauchy's integral Formula we have:

$$f(z) = \frac{1}{2\pi i}\oint_{C1} \frac{f(z^*)}{(z^*-z)}\,dz^* - \frac{1}{2\pi i}\oint_{C2} \frac{f(z^*)}{(z^*-z)}\,dz^*$$

For the 2nd integral we note that

$$\left|\frac{z^*-z_0}{z-z_0}\right| < 1$$

z is in the annulus, outside C_2
z^* is on C_2



$$\frac{1}{z^*-z} = \frac{1}{(z^*-z_0)-(z-z_0)} = \frac{-1}{(z-z_0)(1-(z^*-z_0)/(z-z_0))}$$

$$= \frac{-1}{(z-z_0)}\left\{1 + \frac{z^*-z_0}{z-z_0} + \left(\frac{z^*-z_0}{z^*-z_0}\right)^2 + \ldots + \left(\frac{z^*-z_0}{z-z_0}\right)^n\right\} - \frac{1}{z-z^*}\left(\frac{(z^*-z_0)}{z-z_0}\right)^{n+1}$$

Multiply by $-f(z^*)/2\pi i$ and integrate over C_2 on both sides gives 2^{nd} integral and series of b_n coefficients as required plus a remainder $R^*_n(z)$ which we can show $=0$ as $n\to\infty$ (skipped)

$$R^*_n(z) = \frac{1}{2\pi i(z-z_0)^{n+1}}\oint_{C2} \frac{(z^*-z_0)^{n+1}}{(z-z^*)}\,f(z^*)dz^*$$

Uniqueness

The Laurent series of a given analytic function f(z) in its annulus of convergence is unique. However, f(z) may have different Laurent series in two annuli with the same center.

Example 1. Find the Laurent series of $z^{-5}\sin z$ with center 0

Solution

$$\text{since } \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \qquad (|z| > 0)$$

$$z^{-5} \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-4}}{(2n+1)!} = z^{-4} - \frac{z^{-2}}{3!} + \frac{1}{5!} - \frac{z^2}{7!} \qquad (|z| > 0)$$

Here the “annulus” of convergence is the whole complex plane without the origin.

Example 2

$f(z) = z^2 e^{1/z}$, find the Laurent series with center 0.

Solution

$$\text{since } e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

replace z by 1/z and multiply by z^2

$$f(z) = z^2 \sum_{n=0}^{\infty} \frac{1}{z^n\, n!} = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} \qquad (|z| > 0)$$

Here the "annulus" of convergence is the whole complex plane without the origin.

Example 3

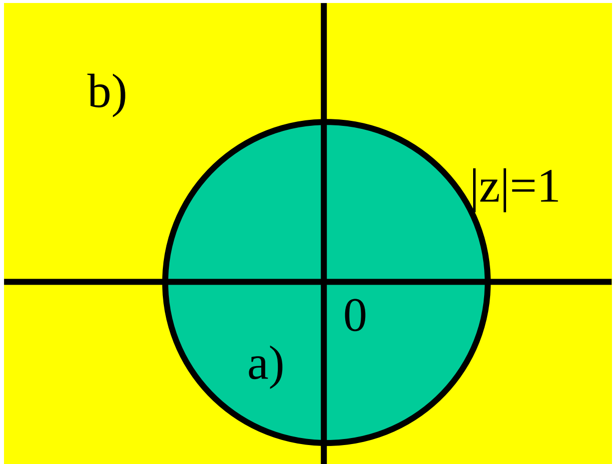
Develop 1/(1-z) a) in non-negative powers of z and
b) in negative powers of z

Solution

a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|< 1)$$

b) $$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \qquad (|z|> 1)$$

Here the annulus of convergence
for a) is different from that of b)



Example 4

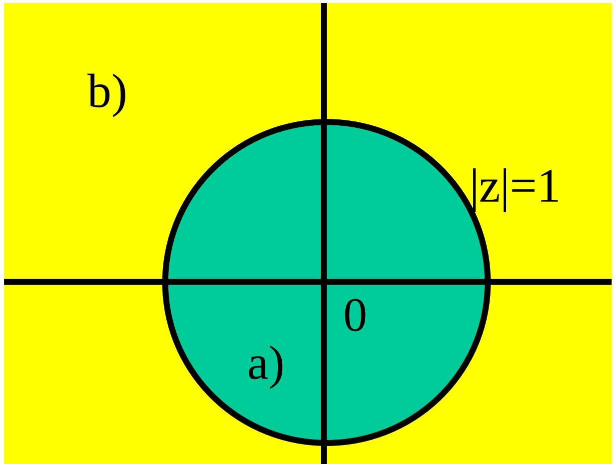
Find all Laurent series of $1/(z^3-z^4)$ with center 0

Solution (similar to previous example, multiply by $1/z^3$)

a) $$\frac{1}{z^3-z^4} = \sum_{n=0}^{\infty} z^{n-3} = z^{-3} + z^{-2} + z + 1 + z + \qquad (0<|z|< 1)$$

b) $$\frac{1}{z^3-z^4} = \frac{-1}{z^4(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \qquad (|z|> 1)$$

Here the annulus of convergence for a) is different from that of b)



Example 5

Find all Taylor and Laurent series of $(3-2z)/(z^2-3z+2)$ with center 0

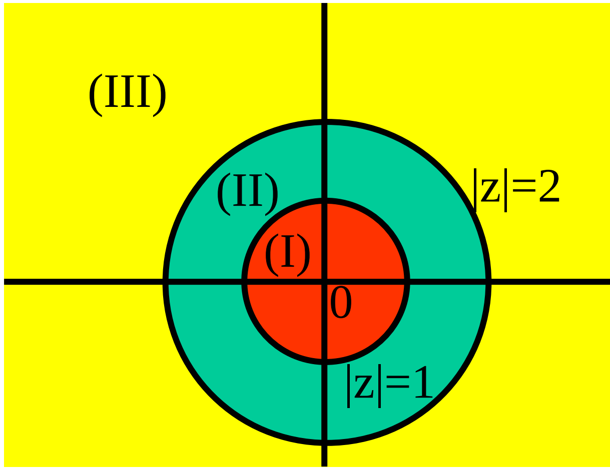
Solution partial fractions give:

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}$$

(a) & (b) in example 3 take care of the 1st fraction. 2nd given by

c) $$\frac{-1}{z-2} = \frac{1}{2(1-z/2)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad (|z|<2)$$

d) $$\frac{-1}{z-2} = \frac{1}{z(1-2/z)} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \quad (|z|>2)$$



(I) from (a) & (c) for ($|z|<1$)

$$f(z) = \sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right)z^n = \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 +$$

(II) from (b) & (c) for ($1<|z|<2$)

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \ldots - \frac{1}{z} - \frac{1}{z^2} -$$

(III) from (d) & (b) for ($|z|>2$)

$$f(z) = \sum_{n=0}^{\infty}(1+2^n)\frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4}$$

Example 6

Find the Laurent series of $1/(1-z^2)$ that converges in the annulus $1/4<|z-1|<1/2$ and determine the precise region of convergence.

Solution The annulus has center 1, so that we must develop

$$f(z) = \frac{-1}{(z-1)(z+1)}$$

in powers of z-1. Since

$$\frac{1}{z+1} = \frac{1}{2+(z-1)} = \frac{1}{2}\frac{1}{[1-(-(z-1)/2)]}$$

$$= \tfrac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n \qquad (|(z-1)/2| < 1)$$

$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(z-1)^{n-1}}{2^{n+1}} = \frac{1}{2(z-1)} + \frac{1}{4} - \frac{(z-1)}{8} - \frac{(z-1)^2}{16} - \ldots\ldots$$

The precise region of convergence is $0<|z-1|<2$