ERG 2012B

Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #8

Power Series II

Functions Given by Power Series

If an arbitrary power series has a non-zero radius of convergence its sum is a function of z, say, f(z).

Without loss of generality, we can take $z_0=0$ and write:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots\ldots \quad (|z| < R) \qquad (3\text{-}1)$$

We say that f(z) is **represented** by the power series or that it is **developed** in the power series

We need to show that this power series representation is **unique** for any function f(z) – in other words *a function f(z) cannot be represented by two different power series with the same center.*

Theorem 3-1

Continuity of the sum of a power series

The function f(z) in (3-1) with R>0 is continuous at z=0

Proof

By definition of continuity we must show

$$\lim_{z\to 0} f(z) = f(0) = a_0$$

The detailed proof is skipped.

Theorem 3-2

Identity theorem for power series

Suppose that the power series

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

both converge for $|z|<R$, where R is positive, and have the same sum for all these z. Then the series must be identical i.e.

$$a_n = b_n \qquad \text{for all } n=0,1.....$$

Proof: If we assume

$$a_0+a_1z+a_2z^2+...= b_0+b_1z+b_2z^2+..... \qquad (|z|<R)$$

The sums of the power series are continuous at $z=0$ (Thm3-1).

Hence $a_0=b_0$ Therefore true for $n=0$

We can now subtract a_0 and b_0 from each side and divide by z

With the new power series formed we can show $a_1=b_1$ in the same way. By repeating this we can show that $a_n=b_n$ for all n

Termwise addition or subtraction

Our ultimate goal is to show that *every* analytic function can be represented by a power series – But first the operations of power series.

Termwise addition or subtraction of two power series with radii of convergence R_1 and R_2 yields a power series with radii of convergence at least equal to the smaller of R_1 and R_2

Proof: Add or subtract the partial sums S_n and S_n^* term by term and use $\lim (S_n \pm S_n^*) = \lim S_n \pm \lim S_n^*$

Termwise Multiplication

Termwise multiplication of two power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \ldots..$$

and $$g(z) = \sum_{m=0}^{\infty} b_m z^m = b_0 + b_1 z + \ldots..$$

means the multiplication of each term of the first series by each term of the second series and the collection of like powers of z.

This gives a power series, called the **Cauchy product** given by

$$a_0b_0+(a_0b_1+a_1b_0)z+(a_0b_2+a_1b_1+a_2b_0)z^2+\ldots$$

$$= \sum_{n=0}^{\infty}(a_0b_n+a_1b_{n-1}+\ldots.\ a_nb_0)z^n$$

This power series converges absolutely for each z within the circle of convergence of each of the two given series and has the sum s(z) = f(z)g(z).

Termwise Differentiation

The series obtained from $\Sigma a_n z^n$ by termwise differentiation is called the **derived series** of the power series

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \ldots..$$

Theorem 3-3 (termwise differentiation of a power series)

The derived series of a power series has the same radius of convergence as the original series

Proof: skip.

Example

Find the radius of convergence R of the following series using Theorem 3

$$\sum_{n=2}^{\infty} C_2^n z^n = z^2 + 3z^3 + 6z^4 + 10z^5 \ldots..$$

where C_s^n are the binomial coefs $= \binom{n}{s} = \dfrac{n(n-1)(n-2)..(n-s+1)}{s!}$

so that $\binom{n}{2} = \dfrac{n(n-1)}{2!}$

Solution

Consider $f(z) = \sum_{n=0}^{\infty} z^n$ then $f^{/}(z) = \sum_{n=1}^{\infty} nz^{n-1}$, $f^{//}(z) = \sum_{n=2}^{\infty} n(n-1)z^{n-2}$

$$\Rightarrow (½)z^2 f^{//}(z) = \sum_{n=2}^{\infty} n(n-1)z^n/2 = \sum_{n=2}^{\infty} C_2^n z^n$$

Since f(z) has radius of convergence = 1 , then R=1 by Thm 3

Theorem 3-4

Termwise integration of Power Series

The power series

$$\sum_{n=0}^{\infty}(a_n z^{n+1})/(n+1) = a_0 z + (a_1/2)z^2 + (a_2/3)z^3 + \ldots.$$

obtained by integrating the series $a_0+a_1z+a_2z^2+$ term by term has the same radius of convergence as the original series

Proof: is similar to that of Thm 3.3

Theorem 3-5

Analytic functions and their derivatives

A power series with a non-zero radius of convergence R represents an analytic function at every point interior to its circle of convergence.

The derivatives of this function are obtained by differentiating the original series term by term.

All the series thus obtained have the same radius of convergence as the original series. Hence each of them represents an analytic function.

Proof: skip.

Summary

This means that power series are wonderful! We can:

- differentiate them term by term
- integrate them term by term
- sum of the series are analytic functions for $R>0$

What else?

- every given analytic function can be represented by a power series.......