ERG 2012B
Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #7
Power Series

Sequence

- An **infinite sequence,** or briefly, a **sequence**, is obtained by assigning to each positive integer *n* a number z_n, called a **term** of the sequence, written as:

  $z_1, z_2, \ldots\ldots\ldots$ or $\{z_1, z_2, \ldots\ldots\}$ or briefly $\{z_n\}$

- A **real sequence** is one of which all terms are real.

Convergence

- A **convergent sequence** $z_1, z_2, \ldots\ldots$ is one that has a limit c, written:

  $\lim_{n \to \infty} z_n = c$ or simply $z_n \to c$

- By definition of limit, this means that given any

  $\varepsilon > 0, \exists N$ s.t.

  $|z_n - c| < \varepsilon \quad \forall n > N \qquad (1)$

- A **divergent sequence** is one that does not converge.

Example

The sequence $\{i^n/n\} = \{i, -1/2, -i/3, 1/4, \ldots\ldots\}$ is convergent with limit 0

The sequence $\{i^n\} = \{i, -1, -i, 1, \ldots.\}$ is divergent

The sequence $\{z_n\}$ with $z_n = (1+i)^n$ is divergent

$= \{ 1+i, 2i, -2+2i, -4, -4-4i, \ldots.\}$

The sequence $\{z_n\}$ with $z_n = 2-1/n + i(1+2/n)$ is convergent

$= \{ 1+3i, 3/2+2i, 5/3+5i/3, 7/4+3i/2, \ldots.\}$

The limit is $c = 2+i$ as $n \to \infty$

(and $|z_n - c| = |-1/n+2i/n| = \sqrt{5}/n < \varepsilon$ if $n > \sqrt{5}/\varepsilon$)

Theorem 1

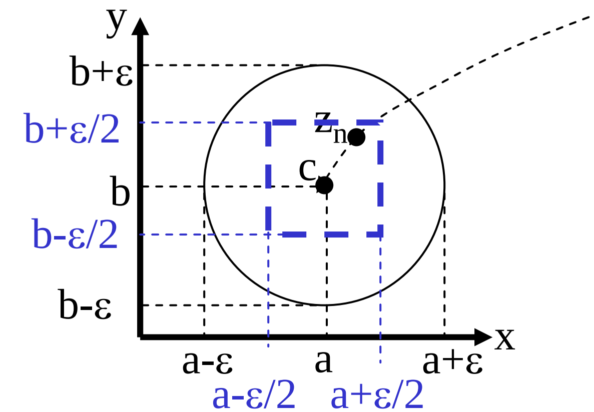
A sequence $\{z_n\}$ of complex numbers $z_n = x_n + iy_n$ converges to $c = a + ib$ iff (if and only if) the sequence of the real parts $\{x_n\}$ converges to a and the sequence of imaginary parts $\{y_n\}$ converges to b

Proof: If $|z_n - c| < \varepsilon$, then $z_n = x_n + iy_n$ is within the circle of radius ε about $c = a + ib$ so that

$$|x_n - a| < \varepsilon,\ |y_n - b| < \varepsilon$$

Hence convergence $z_n \to c$ implies

$$x_n \to a, \text{ and } y_n \to b$$



Conversely, if $x_n \to a$ and $y_n \to b$ as $n \to \infty$ then for a given $\varepsilon > 0$, we can choose N sufficiently large that $\forall\ n > N$

$$|x_n - a| < \varepsilon/2,\ |y_n - b| < \varepsilon/2$$

so that $z_n = x_n + iy_n$ lies in a square with center c and side ε.

Hence z_n lies within a circle of radius ε with center c

Series

Given a sequence $\{z_n\}$, an **infinite series**, or **series** can be formed by the infinite sum:

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \ldots.. \qquad (2)$$

The z_1, z_2, ... are called the **terms** of the series.

The sequence of sums:

$$\begin{aligned} s_1 &= z_1 \\ s_2 &= z_1 + z_2 \\ s_3 &= z_1 + z_2 + z_3 \qquad (3) \\ &\ldots\ldots.. \\ s_n &= z_1 + z_2 + \ldots.z_n \end{aligned}$$

are the sequence of **partial sums** of the infinite series.

Convergent Series

A **convergent series** is one of which the sequence of partial sums converges, i.e.

$$\lim_{n\to\infty} s_n = s$$

where s is called the **sum** or **value** of the series, written:

$$s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \ldots..$$

A **divergent series** is one that does not converge

The **remainder** of the series (2) after the term z_n is

$$R_n = z_{n+1} + z_{n+2} + z_{n+3} + \ldots..$$

If (2) converges and has the sum s, then

$$s = s_n + R_n \quad \text{or} \quad R_n = s - s_n \quad \text{and } R_n \to 0$$

Theorem 2

A series with $z_m=x_m+iy_m$ converges with sum $s=u+iv$ iff

$x_1+x_2+\ldots..$ converges with the sum u and $y_1+y_2+\ldots..$ converges with the sum v

Proof: by application of theorem 1 to the partial sums.

Theorem 3

If a series $z_1+z_2+\ldots.$ converges, then $\lim_{m\to\infty} z_m=0$. Hence if this does not hold, the series diverges.

Proof: If $z_1+z_2+\ldots$ converges with the sum s, then, since

$$z_m=s_m - s_{m-1},$$

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} (s_m-s_{m-1}) = \lim_{m\to\infty} s_m - \lim_{m\to\infty} s_{m-1} = s-s = 0$$

- $z_m\to 0$ is necessary for convergence but **not sufficient**
  e.g. the harmonic series $1+1/2+1/3+\ldots$
  $z_m \to 0$ **but** the series diverges.

Theorem 4

Cauchy's convergence principle for series.

A series $z_1+z_2+....$ is convergent iff given any $\varepsilon>0$ we can find an N s.t.

$$|z_{n+1}+z_{n+2}+...+z_{n+p}| < \varepsilon \quad \text{for every } n> N \text{ and } p=1,2...$$

Proof: Skipped.

Absolute Convergence: a series $z_1+z_2+...$ is called **absolutely convergent** if the series of the absolute values of the terms

$$\sum_{m=1}^{\infty}|z_m| = |z_1| + |z_2| + ...$$

is convergent

- If $z_1+z_2+...$ converges but $|z_1|+|z_2|+....$ diverges, then the series z_1+z_2 .... is called **conditionally convergent**
- If a series is absolutely convergent it is convergent

Example The series 1-1/2+1/3-1/4+... converges conditionally

Theorem 5

Comparison Test.

If a series $z_1+z_2+\ldots$ is given and we can find a converging series $b_1+b_2+\ldots$ with non negative real terms s.t. $|z_n| \le b_n$ for every $n=1,2\ldots$

Then the given series converges, even absolutely

Proof: By Cauchy's principle, since $b_1+b_2+\ldots$ converges, for any given $\varepsilon>0$, we can find an N s.t. $b_{n+1}+\ldots+b_{n+p} < \varepsilon$ for every $n>N$ and $p=1,2,\ldots$.

So $|z_1|\le b_1$, $|z_2| \le b_2,\ldots\ldots$ hence $|z_{n+1}|+\ldots.|z_{n+p}|\le b_{n+1}+\ldots.b_{n+p}< \varepsilon$

Hence again by Cauchy's principle, $|z_1|+|z_2|+\ldots$ converges so that $z_1+z_2+\ldots$ is absolutely convergent

Theorem 6

Geometric Series

The geometric series

$$\sum_{m=0}^{\infty} q^m = 1+q+q^2+\ldots$$

converges with the sum 1/(1-q) if |q|<1 and diverges otherwise

Proof:

If $|q| \geq 1$ then $|q^m| \geq 1$ and theorem 3 implies divergence.

Now let $|q| < 1$, then the n^{th} partial sum is

$$s_n = 1+q+\ldots+q^n$$

and

$$q\, s_n = q+\ldots+q^n+q^{n+1}$$

so $(1-q)s_n = 1-q^{n+1}$

and $s_n = (1-q^{n+1)}/(1-q) = [1/(1-q)] - [q^{n+1}/(1-q)]$ as $(1-q)\neq 0$

as $|q|<1$, $q^{n+1}/(1-q) \to 0$ as $n \to \infty$

$\therefore$ series is convergent and the has the sum 1/(1-q)

Theorem 7

Ratio Test

If a series $z_1+z_2+\ldots.$ with $z_n \neq 0$ $(n=1,2,..)$ has the property that

$$|z_{n+1}/z_n| \leq q < 1 \quad (\forall\ n > N,\ q \text{ fixed, } N \text{ arbitrary})$$

The series converges absolutely. However, if

$$|z_{n+1}/z_n| \geq 1\ (\forall\ n > N) \text{ the series diverges}$$

Proof:

If $|z_{n+1}/z_n| \geq 1$ then $|z_{n+1}| \geq |z_n|$ for $n>N \Rightarrow$ divergence (Thm 3)

If $|z_{n+1}/z_n| \leq q$ then $|z_{n+1}| \leq |z_n|q$ for $n>N$, & $|z_{N+p}| \leq |z_{N+1}|q^{p-1}$

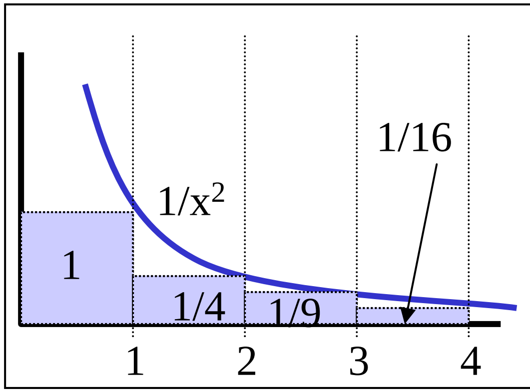
Hence $|z_{N+1}|+|z_{N+2}|+|z_{N+3}|+\ldots \leq |z_{N+1}|(1+q+q^2+\ldots)$

$\therefore$ $z_1+z_2+\ldots\ldots$ is absolutely convergent by comparison with the geometric series (Thm 5 & 6)

Theorem 8

Ratio Test

If a series $z_1+z_2+\ldots.$ with $z_n \neq 0$ $(n=1,2,..)$ is s.t.



$$\lim_{n\to\infty} |z_{n+1}/z_n| = L$$

Then we have the following:

a) If $L < 1$ the series converges absolutely

b) If $L > 1$ it diverges

c) If $L = 1$, the test fails – no conclusion possible

Proof: (a) as n increases $|z_{n+1}/z_n|$ gets closer to L (<1) so that

$\exists$ N s.t. $|z_{n+1}/z_n| < 1 \ \forall \ n > N$ so series converges absolutely

(b) similarly if $L > 1$ the series diverges

(c) Consider the harmonic series $1+1/2+1/3+\ldots$ $|z_n/z_{n+1}| \to 1$ and it diverges.

But $1+1/4+1/9+\ldots.$ also has $|z_n/z_{n+1}| = n^2/(n+1)^2 \to 1$ and it converges:

$S_n = 1+1/4+..+1/n^2 \leq 1+\int_1^n (1/x^2)dx = 2-1/n$;

S_n is monotonic increasing and bounded above so it converges

Example

Given $S = \sum_{0}^{\infty} \frac{(100+75i)^n}{n!} = 1+(100+75i)+(100+75i)^2/2!+\ldots$

Is S convergent or divergent?

Soln: $\left|\frac{z_{n+1}}{z_n}\right| = \frac{|100+75i|^{n+1}/(n+1)!}{|100+75i|^n/n!} = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \to L=0$

$\therefore$ by theorem 8 the series is convergent.

Example II

Given $S = \sum_{0}^{\infty} \left(\frac{i}{2^{3n}} + \frac{1}{2^{3n+1}}\right) = i + 1/2 + i/8 + 1/16 + i/64 + 1/128 + \ldots$

Is S convergent?

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + \ldots$$

Soln: If we take S as $z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + \ldots$

then $\left|\frac{z_{n+1}}{z_n}\right|$ is either ½ or ¼

Thm 8 is not applicable

but ratio is ≤ ½ < 1 so convergent by Thm 7

But if we take $z_n = \frac{i}{2^{3n}} + \frac{1}{2^{3n+1}}$ then

then $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{2^{-3(n+1)}(i+½)}{2^{-3n}(i+½)}\right| = 2^{-3} = 1/8 < 1$

is convergent by Thm 7 or Thm 8

Theorems 7 & 8

- Thm 7 is more general than Thm 8 since Thm 7 does not require the ratio to have a limit

- In application of Thm 7, note that the requirement

$$\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1 \ \text{ implies } \left|\frac{z_{n+1}}{z_n}\right| < 1$$

but $|z_{n+1}/z_n| < 1$ does not imply convergence

– only if < 1 as $n \to \infty$

For harmonic series $|z_{n+1}/z_n| = n/(n+1) < 1 \ \forall \ n$ but is divergent

and $n/(n+1) \to 1$ as $n \to \infty$

Theorem 9

Root Test

If a series $z_1+z_2+\ldots$ is such that for every n > some N

$$\sqrt[n]{|z_n|} \le q < 1 \qquad (n < N) \qquad (*)$$

(where q<1 is fixed), the series converges absolutely. If for infinitely many n

$$\sqrt[n]{|z_n|} \ge 1 \qquad (**)$$

the series diverges

Proof: If (*) holds, then $|z_n| \le q^n < 1 \quad \forall\ n > N$

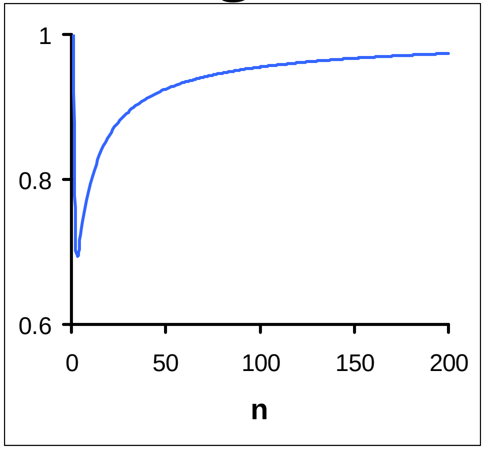
Hence $z_1+z_2+\ldots$ converges absolutely by comparison with the geometric series. If (**) holds, then $|z_n| \ge 1$ for infinitely many n – by Thm 3 this series will diverge

Caution: (*) implies $\sqrt[n]{|z_n|} < 1$.

This does not imply convergence.

Harmonic series $\sqrt[n]{(1/n)} < 1$ but divergent

as $n\to\infty$, $\sqrt[n]{(1/n)} \to 1$



Theorem 10

Root Test

If a series $z_1+z_2+\ldots$ is such that for every n > some N

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L \qquad (***)$$

then we have the following:

a) If L < 1 the series converges absolutely
b) If L > 1 it diverges
c) If L = 1 the test fails; i.e. no conclusion is possible

Proof: (a) Let L=1-a* < 1 then for some sufficiently large N*

$\sqrt[n]{|z_n|} < q = 1\text{-}a^*/2 < 1 \quad \forall\, n > N^*$

Hence $|z_n|<q^n<1 \;\; \forall\, n > N^* \Rightarrow$ absolute convergence

(b) If L > 1 $\sqrt[n]{|z_n|} > 1 \;\; \forall$ n sufficiently large

Hence $|z_n| > 1$ for those n $\Rightarrow$ divergence by Thm 3

(c) Both divergent harmonic series and convergent $1/n^2$ give L=1

Example

Given $S = \sum_{0}^{\infty} \left(\frac{(-1)^n}{2^{2n} + 3}\right) (4-i)^n = \frac{1}{4} - \frac{1}{7}(4-i) + \frac{1}{19}(4-i)^2 - \ldots$

Is S convergent?

$$\sqrt[n]{(|(4-i)^n|/(2^{2n}+3))} = |4-i| \,/\, (\sqrt[n]{(4^n+3)})$$

$$= \sqrt{17} \,/\, (\sqrt[n]{(4^n+3)}) \rightarrow \sqrt{17} \,/\, 4 > 1$$

$\therefore$ S diverges by Thm 10

Power Series

Definition

A **power series** in powers of $z\text{-}z_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (1)$$

where z is a variable, a_0, a_1, ... are constants, called coefficients of the series and z_0 is a constant called the center of the series

A power series in powers of z is a particular case when $z_0=0$

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (2)$$

Convergence Behaviour

Example 1

The geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2$$

converges absolutely if $|z| < 1$ and diverges if $|z| \geq 1$ (by Thm 6)

Example 2

The power series

$$\sum_{n=0}^{\infty} z^n/n! = 1 + z + z^2/2 + z^3/6 + \ldots.$$

converges absolutely for every z.

By the ratio test for any fixed z

$$\left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \frac{|z|}{n+1} \to 0 \qquad \text{as } n \to \infty$$

Example 3

The power series

$$\sum_{n=0}^{\infty} n!z^n = 1 + z + 2z^2 + 6z^3 + \ldots..$$

converges only for $z=0$ and diverges for every $z\neq 0$.

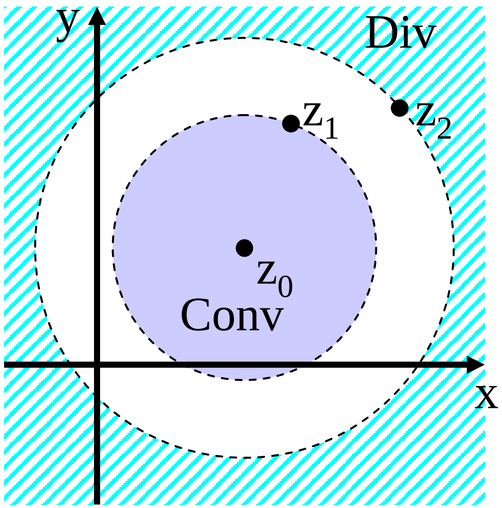
Again by the ratio test

$$\left|\frac{z^{n+1}(n+1)!}{z^n\, n!}\right| = (n+1)\,|z| \to \infty \quad \text{as } n\to\infty \quad (z \text{ fixed and} \neq 0)$$

Theorem 2-1

a) If the power series (1) converges at a point $z=z_1\neq z_0$, it converges absolutely for every z closer to z_0 then z_1, i.e. $|z-z_0| < |z_1-z_0|$

b) If (1) diverges at a $z = z_2$, it diverges for every z farther away from z_0 than z_2, i.e. $|z-z_0| > |z_2-z_0|$



Proof:

(a) Since the series converges for z_1, Thm 3 tells us $a_n(z_1-z_0)^n \to 0$ as $n\to\infty$

$\Rightarrow |a_n(z_1-z_0)^n| < M$ for every $n=0,1,\ldots.$

$$\Rightarrow |a_n(z-z_0)^n| = |a_n(z_1-z_0)^n((z-z_0)/(z_1-z_0))^n|$$
$$\leq M|(z-z_0)/(z_1-z_0)|^n$$

Now $|z-z_0| < |z_1-z_0|, \Rightarrow |(z-z_0)/(z_1-z_0)| < 1$

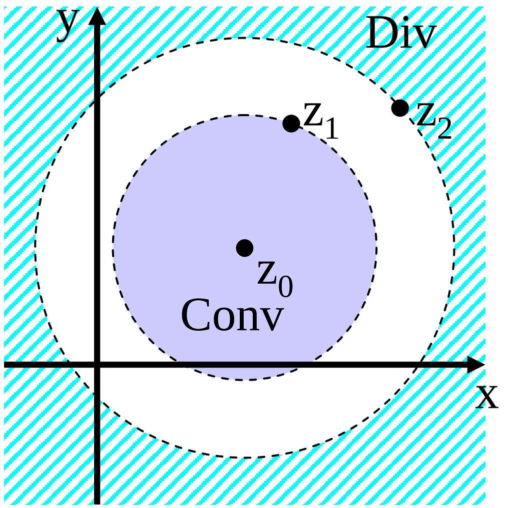
Hence the series $M\sum_0^\infty|(z-z_0)/(z_1-z_0)|^n$ is a converging geometric series (Thm 6)

Hence the series $\sum_0^\infty a_n(z_1-z_0)^n$ is absolutely convergent by comparison for $|z-z_0| < |z_1-z_0|$

Theorem 2-1

a) If the power series (1) converges at a point $z=z_1 \neq z_0$, it converges absolutely for every z closer to z_0 then z_1, i.e. $|z-z_0| < |z_1-z_0|$

b) If (1) diverges at a $z = z_2$, it diverges for every z farther away from z_0 than z_2, i.e. $|z-z_0| > |z_2-z_0|$



Proof:

(b) If the statement is false we would be able to find convergence at some point z_3 s.t. $|z_3-z_0|>|z_2-z_0|$

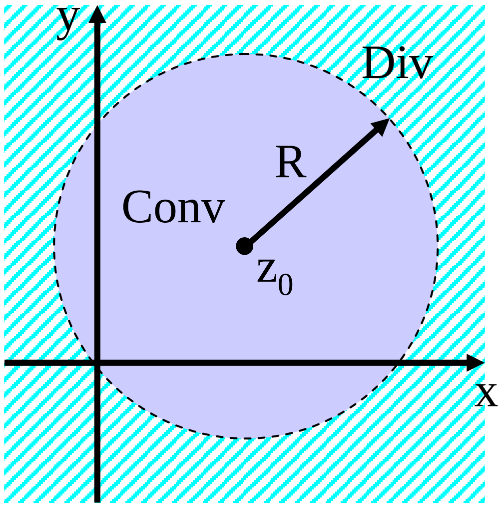
But then this implies that we should also have convergence at z_2 by the results of (a)

Thus either z doesn't diverge at z_2 after all *or* there is no convergence at any z_3

Radius of Convergence

Let R be the radius of the **smallest** circle with centre z_0 that includes all the points at which the power series (1) converges. Then (1) is convergent for all z s.t. $|z-z_0| < R$ and divergent for all z s.t. $|z-z_0| > R$

The circle $|z-z_0| = R$ is called the **circle of convergence** and its radius R **the radius of convergence**



- We write $R=\infty$ if the series converges for all z
- We write $R=0$ if the series converges only at $z=z_0$
- No general statements can be made about the convergence of a power series on the circle of convergence itself

Example

The radius of convergence is R=1 for the series:

$$\Sigma z^n/n^2\ ,\ \Sigma z^n/n \text{ and } \Sigma z^n$$

On the circle of convergence

- $\Sigma z^n/n^2$ converges everywhere since $\Sigma 1/n^2$ converges
- $\Sigma z^n/n$ converges at -1 but diverges at 1
- Σz^n diverges everywhere

Theorem 2-2

If the sequence $|a_{n+1}/a_n|$, n=1,2,.... converges with limit L^* then

If $L^*=0$ then $R=\infty$, i.e. the power series converges for all z

If $L^*\neq 0$ ($L^*>0$) then $R=1/L^*$ (Cauchy-Hadamard formula)

If $L^*=\infty$ then R=0

Proof: The series (1) has the terms $z_n=a_n(z-z_0)^n$. From the ratio test (Thm 8)

$$L=\lim_{n\to\infty}|z_{n+1}/z_n|=\lim_{n\to\infty}|a_{n+1}(z-z_0)^{n+1}/a_n(z-z_0)^n|=\lim_{n\to\infty}|a_{n+1}/a_n||z-z_0|$$

or $L=L^*|z-z_0|$

- If $L^*=0$, then $L=0\ \forall\ z$, ratio test gives convergence $\forall\ z$
- If $L^*>0$ and $|z-z_0|<1/L^*, \Rightarrow L=L^*|z-z_0|<1$ $\therefore$ convergent

  If $L^*>0$ and $|z-z0|>1/L^*$, then $L>1$ $\therefore$ divergent

  Hence, by definition, $1/L^*$ is the radius of convergence R
- If $L^*=\infty$ then $|z_{n+1}/z_n|\geq 1\ \forall\ z\neq z_0$ $\therefore$ divergent $\forall\ z\neq z_0$

Example

Determine the radius of convergence R of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} (z-3i)^n$$

Solution:

$$L^* = \lim_{n\to\infty} \frac{(2n+2)!/[(n+1)!]^2}{(2n)!/(n!)^2} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4$$

$$\therefore\ R = 1/L^* = 1/4$$

The series converges in the open disk $|z-3i| < ¼$

Example 2

Find the radius of convergence R of the power series

$$\sum_{n=0}^{\infty} [1+(-1)^n+1/2^n]z^n = 3+2^{-1}z+(2+2^{-2})z^2+2^{-3}z^3+(2+2^{-4})z^4+\ldots$$

Solution:

the sequence of the ratio $|a_{n+1}/a_n|$ is $1/6,\ 2(2+2^{-2}),\ 1/(2^3(2+2^{-2}))$

does not converge. Thm 2-2 can not be applied

If we use the root test in Thm 2-2 instead of the ratio test we get: $R = 1/\tilde{L}$, $\quad \tilde{L} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (5*)

But $\sqrt[n]{|a_n|}$ does not converge but has 2 limit points ½ and 1

For odd n's, $\tilde{L} = \lim \sqrt[n]{|1/2^n|} = ½$

For even n's, $\tilde{L} = \lim \sqrt[n]{|2+1/2^n|} = 1$

It can be shown that $R= 1/\tilde{L}$, where $\tilde{L}$ is the greatest limit point

So that $\tilde{L}=1$ hence $R = 1$ $\quad\therefore$ series converges for $|z| < 1$