ERG 2012B
Advanced Engineering Mathematics II

Part I: Complex Variables

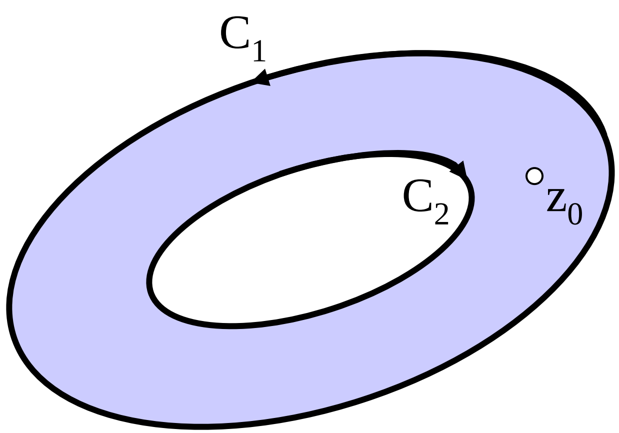
Lecture #6
Derivatives of Analytic Functions

Multiply Connected Domains

If f(z) is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2, and z_0 is any point in that domain, then:

$$f(z_0) = 1/(2\pi i)\oint_{C_1} f(z)/(z-z_0)\,dz + 1/(2\pi i)\oint_{C_2} f(z)/(z-z_0)\,dz$$

where the outer integral (over C_1) is taken ↺ and the inner ↻



Derivatives of Analytic Functions

Theorem: If f(z) is analytic in a domain D, then it has derivatives of all orders in D, which are also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas:

(1′) $$f'(z_0) = 1/(2\pi i) \oint_C f(z)/(z-z_0)^2\, dz$$

(1″) $$f''(z_0) = 2!/(2\pi i) \oint_C f(z)/(z-z_0)^3\, dz$$

and in general:

(1) $$f^{(n)}(z_0) = n!/(2\pi i) \oint_C f(z)/(z-z_0)^{n+1}\, dz$$

where C is any simply connected closed path in D that encloses z_0 and whose full interior belongs to D and we integrate ↻ around C.

Derivatives of Analytic Functions

Proof: To prove (1′) we start from the definition

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta F}{\Delta z}$$

Cauchy's integral formula: $f(z_0) = 1/(2\pi i) \oint_C f(z)/(z-z_0)dz$

$$\frac{\Delta F}{\Delta z} = \frac{1}{2\pi i\, \Delta z}\left[\oint_C \frac{f(z)}{z-(z_0+\Delta z)}\, dz - \oint \frac{f(z)}{z-z_0} dz\right]$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\, dz \underset{\Delta z \to 0}{\longrightarrow} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz$$

The limit of $\Delta F/\Delta z$ exists as $\Delta z \to 0$. Hence (1′) is proved. Similarly, (1″) can be proved and by induction, (1) can be proved.

Alternative Formulas

(1′) $$f'(z_0) = 1/(2\pi i) \oint_C f(z)/(z-z_0)^2 \, dz$$

$$\oint_C \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0)$$

(1″) $$f''(z_0) = 2!/(2\pi i) \oint_C f(z)/(z-z_0)^3 \, dz$$

$$\oint_C \frac{f(z)}{(z-z_0)^3} dz = \frac{2\pi i}{2!} f''(z_0)$$

(1) $$f^{(n)}(z_0) = n!/(2\pi i) \oint_C f(z)/(z-z_0)^{n+1} \, dz$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Examples

For any contour enclosing the point πi (counterclockwise)

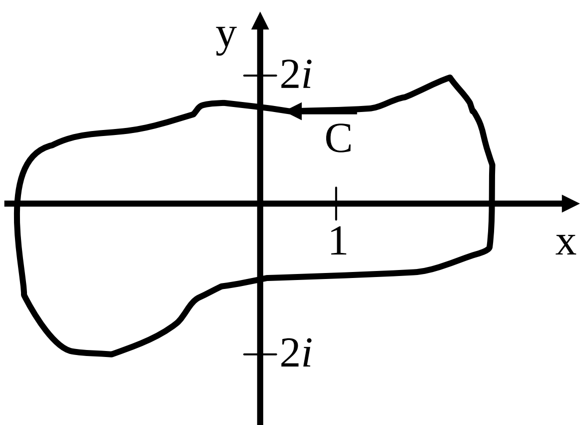
$$\oint_C \cos(z)/(z-\pi i)^2 dz = 2\pi i(\cos z)'\Big|_{z=\pi i} = -2\pi i \sin(\pi i) = 2\pi \sinh(\pi)$$

For any contour enclosing the point $-i$ (counterclockwise)

$$\oint_C (z^4-3z^2+z)/(z+i)^3 dz = \pi i(z^4-3z^2+z)''\Big|_{z=-i} = \pi i[12z^2-6]_{z=-i} = -18\pi i$$

For any contour for which 1 lies inside and $\pm 2i$ lie outside ↺

$$\oint_C e^z/[(z-1)^2(z^2+4)]dz = 2\pi i(e^z/(z^2+4))'\Big|_{z=1} = 2\pi i e^z((z^2+4)-2z)/(z^2+4)^2\Big|_{z=1}$$



$$=6e\pi i/25 \approx 2.050\, i$$

Morera's Theorem

If f(z) is continuous in a simply connected domain D and if $\oint_C f(z)dz=0$ for every closed path in D, then f(z) is analytic in D

Proof:

If $\oint_C f(z)dz = 0$, $F(z) = \int_{z_0}^{z} f(z^*)dz^*$ can be defined since the integral is independent of path. From proof of indefinite integral

$$F'(z) = f(z) \quad \forall\ z \text{ in } D \qquad \therefore\ F(z) \text{ is analytic}$$

and therefore $F'(z)$ is also analytic in D.

Cauchy's Inequality

If we choose the contour C to be a circle of radius *r* and center at z_0 and apply the ML inequality to the expression for n^{th} derivative of an analytic function, if $|f(z)| \le M$ on C, then

$$|f^{(n)}(z_0)| = (n!/2\pi)\left|\oint_C f(z)/(z-z_0)^{n+1}dz\right| \le (n!/2\pi)(M/r^{n+1})2\pi r$$

or

$$\left|f^{(n)}(z_0)\right| \le n!\frac{M}{r^n}$$

Liouville's Theorem

If an entire function f(z) is bounded in absolute value for all z, then f(z) must be a constant.

Proof

by assumption $|f(z)| < K$ for all z

by Cauchy's inequality $|f'(z)| < K/r$

Since f(z) is entire, this is true for every r.

We can take r as large as we wish. Hence $f'(z_0) = 0$

Since z_0 is arbitrary, hence $f'(z) = 0$ for all z

and so f(z) is constant.

Summary

For arithmetic operations with **complex numbers**

(1) $$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex

Solutions of Laplace's equation having *continuous* second-order partial derivatives are called *harmonic functions*. The real and imaginary parts of an analytic function are harmonic functions.

If $f(z)$ is analytic in D, then $u(x, y)$ and $v(x, y)$ satisfy the (very important!) **Cauchy–Riemann equations** (Sec. 12.4)

(3) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

everywhere in D. Then u and v also satisfy **Laplace's equation**

(4) $$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

everywhere in D. If $u(x, y)$ and $v(x, y)$ are continuous and have *continuous* partial derivatives in D that satisfy (3) in D, then $f(z) = u(x, y) + iv(x, y)$ is analytic in D. See Sec. 12.4. (More on Laplace's equation and complex analysis follows in Chap. 16.)

Summary

The complex **exponential function** (Sec. 12.6)

$$e^z = \exp z = e^x (\cos y + i \sin y) \tag{5}$$

reduces to e^x if $z = x$ $(y = 0)$. It is periodic with $2\pi i$ and has the derivative e^z.

The **trigonometric functions** are (Sec. 12.7)

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y$$

(6)

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y$$

$\tan z = (\sin z)/\cos z$, $\cot z = 1/\tan z$, etc.

Summary

The **hyperbolic functions** are (Sec. 12.7)

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) = \cos iz,$$
(7)
$$\sinh z = \frac{1}{2}(e^z - e^{-z}) = -i \sin iz,$$

etc. An **entire function** is a function that is analytic everywhere in the complex plane. The functions in (5)–(7) are entire.

The **natural logarithm** is (Sec. 12.8)

$$\ln z = \ln |z| + i \arg z \qquad (\arg z = \theta,\ z \neq 0)$$
(8)
$$= \ln |z| + i \operatorname{Arg} z \pm 2n\pi i \qquad (n = 0, 1, \cdots),$$

where $\operatorname{Arg} z$ is the **principal value** of $\arg z$, that is, $-\pi < \operatorname{Arg} z \leqq \pi$. We see that $\ln z$ is infinitely many-valued. Taking $n = 0$ gives the **principal value** $\operatorname{Ln} z$ of $\ln z$; thus

(8*) $$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z.$$

General powers are defined by (Sec. 12.8)

(9) $$z^c = e^{c \ln z} \qquad (c \text{ complex},\ z \neq 0).$$

Summary

The **complex line integral** of a function $f(z)$ taken over a path C is denoted by

$$(1) \qquad \int_C f(z)\,dz \qquad \text{or, if } C \text{ is closed, also by} \qquad \oint_C f(z)\,dz. \qquad \text{(Sec. 13.1)}$$

If $f(z)$ is analytic in a simply connected domain D, then we can evaluate (1) as in calculus by indefinite integration and substitution of limits, that is,

$$(2) \qquad \int_C f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]$$

for every path C in D from a point z_0 to a point z_1 (See Sec. 13.1). These assumptions imply **independence of path,** that is, (2) depends only on z_0 and z_1 (and on $f(z)$, of course) but not on the choice of C (Sec. 13.2). The existence of an $F(z)$ such that $F'(z) = f(z)$ is proved in Sec. 13.2 by Cauchy's integral theorem (see below).

A general method of integration, not restricted to analytic functions, uses the equation $z = z(t)$ of C, where $a \leqq t \leqq b$,

$$(3) \qquad \int_C f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

Summary

Cauchy's integral theorem is the most important theorem in this chapter. It states that if $f(z)$ is analytic in a simply connected domain D, then for every closed path C in D (Sec. 13.2),

$$\oint_C f(z)\, dz = 0. \tag{4}$$

Under the same assumptions and for any z_0 in D and closed path C in D containing z_0 in its interior we also have **Cauchy's integral formula**

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{5}$$

Furthermore, then $f(z)$ has derivatives of all orders in D that are themselves analytic functions in D and (Sec. 13.4)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 1, 2, \cdots). \tag{6}$$

This implies *Morera's theorem* (the converse of Cauchy's integral theorem) and *Cauchy's inequality* (Sec. 13.4)