ERG 2012B

Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #5:

Complex Integration

Complex Integration

- *Complex* definite integrals are called (complex) **line integrals**. They are written

$$\int_C f(z)\,dz$$

- Here the integrand f(z) is integrated over a given curve C in the complex plane, called the **path of integration**

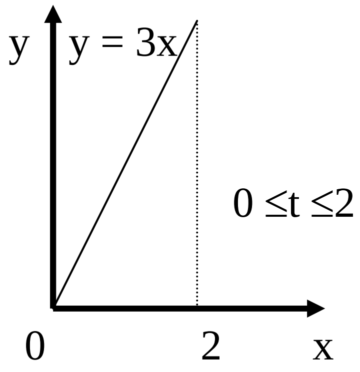
- Such a curve can be represented in the form

$$z(t) = x(t) + i\,y(t) \qquad \text{where t is a real parameter}$$

For example, $z(t) = t + 3it\ ,\ \ 0 \le t \le 2$
represents a portion of the straight line

$$y = 3x$$

and $z(t) = 4\cos t + 4i\sin t\ ,\ \ -\pi \le t \le \pi$
represents the circle $|z| = 4$

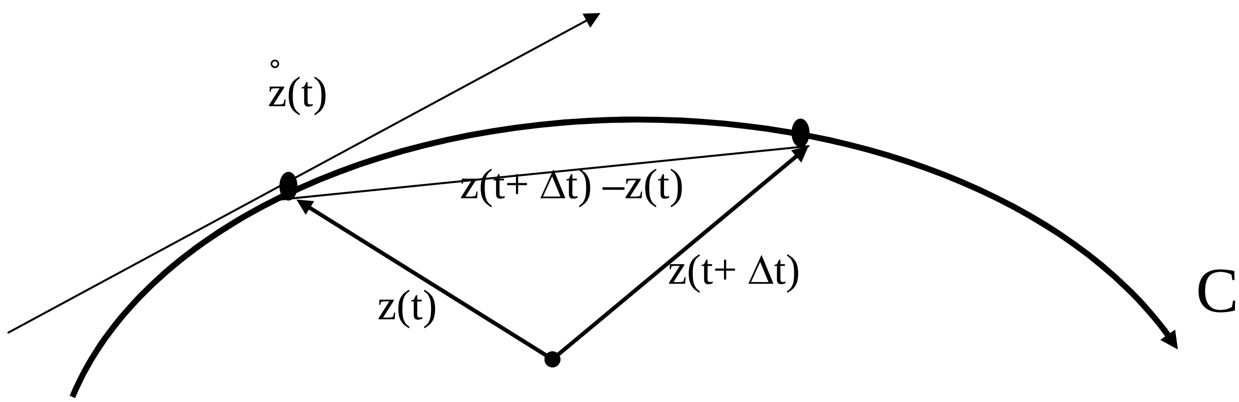


Complex Integration

- C is called a **smooth curve** if it has a continuous and non-zero derivative at each point:

$$\mathring{z}(t) = dz/dt = \mathring{x}(t) + i\,\mathring{y}(t) = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

Geometrically, this means that C has a continuous turning point everywhere



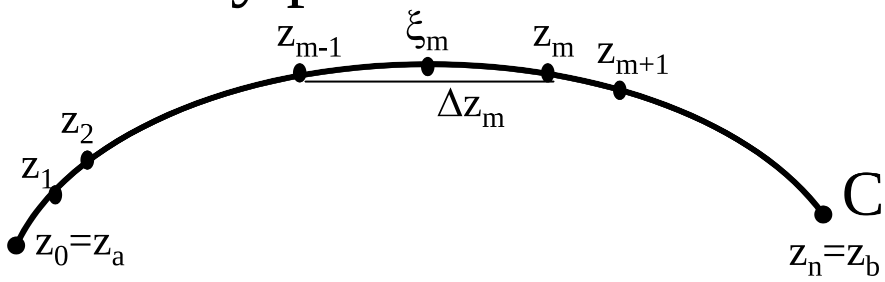
Definition of Complex Line Integral

Consider a smooth curve C in the complex plane given by

$$z(t) = x(t) + i\, y(t) \qquad\qquad a \le t \le b$$

Subdivide (partition) the interval $a \le t \le b$ by points

$$t_0 = a,\ t_1,\ t_2,\ \ldots\ldots,\ t_{n-1},\ t_n = b$$



Corresponding to points on C

$$z_0 = z_a,\ z_1,\ z_2,\ \ldots\ldots,\ z_{n-1},\ z_n = z_b \quad (z_i = z(t_i))$$

On each portion of sub division of C choose an arbitrary point

$$\xi_j = z(t) \quad \text{where} \quad t_j \le t \le t_{j+1}$$

and form the sum

$$S_n = \sum_{m=1}^{n} f(\xi_m)\, \Delta z_m \qquad \text{where } \Delta z_m = z_m - z_{m-1}$$

$$\text{such that } \lim_{n\to\infty} \max|\Delta t_m| = 0$$

$$\text{and hence } \lim_{n\to\infty} |\Delta z_m| = 0$$

The Line Integral

- The limit of the Sum S_n as $n \to \infty$ is called the **line integral** of f(z) over the oriented curve C and is denoted by:

$$\int_C f(z)\, dz$$

- If C is a closed path, it is denoted by:

$$\oint_C f(z)\, dz$$

General Assumption. All paths of integration for complex line integrals are **piecewise smooth**, i.e. they consist of finitely many smooth curves joined end to end.

- From the assumption that f(z) is continuous and C is piecewise smooth it is straightforward to show that the complex line integral exists.

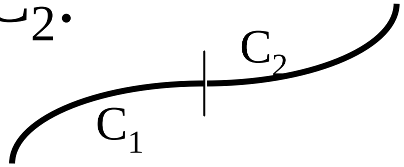
Three Basic Properties

1) Integration is a ***linear*** operation:

$$\int_C [k_1 f_1(z)+k_2 f_2(z)]\, dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz$$

2) If C is ***partitioned*** into two portions C_1 and C_2:

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$



3) ***Sense of reversal***

$$\int_{z_0}^{z_1} f(z)\, dz = - \int_{z_1}^{z_0} f(z)dz$$

where z_0 and z_1 are the end points of the same path C

Theorem: Let C be a piecewise smooth path, represented by $z=z(t)$, where $a \le t \le b$. Let $f(z)$ be continuous on C. Then

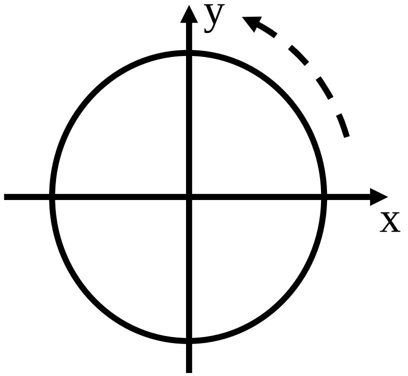
$$\int_C f(z)dz = \int_a^b f[z(t)]\, (dz/dt)\, dt$$

(the proof is straightforward)

Example

Show that $\oint_C (1/z)dz = 2\pi i$ where C is the unit circle counterclockwise

Solution. The unit circle can be represented by



$$z(t) = \cos t + i \sin t \qquad 0 \le t \le 2\pi$$

t: $0 \rightarrow 2\pi \;\Leftrightarrow\;$ counterclockwise

$$dz(t)/dt = -\sin t + i \cos t$$

$f[z(t)] = 1/z(t)$

$$\oint_C (1/z)dz = \int_0^{2\pi} [1/(\cos t + i \sin t)](-\sin t + i \cos t)\, dt = i \int_0^{2\pi} dt = 2\pi i$$

OR

The unit circle can be represented by

$z(t) = e^{it}$, then $1/z(t) = e^{-it}$, $dz = ie^{it}\, dt$

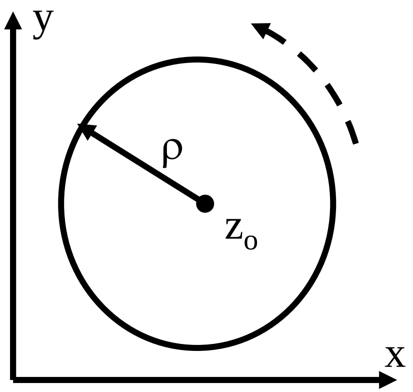
$$\oint_C (1/z)dz = \int_0^{2\pi} e^{-it}\, ie^{it}\, dt = i \int_0^{2\pi} dt = 2\pi i$$

Euler's formula

Example II

Let $f(z) = (z-z_o)^m$ where m is an integer and z_o a constant

Integrate counterclockwise around the circle C of radius ρ with center at z_o



Solution C is represented by

$$z(t) = z_o + \rho e^{it}, \qquad 0 \le t \le 2\pi$$

then $(z-z_o)^m = \rho^m e^{imt}$, $\quad dz = i\rho e^{it}\, dt$

$$I = \oint_C (z-z_o)^m dz = \int_0^{2\pi} \rho^m e^{imt}\, i\rho e^{it} dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t}\, dt$$

$$= i\rho^{m+1}\left[\int_0^{2\pi} \cos(m+1)t\, dt + i\int_0^{2\pi} \sin(m+1)t\, dt\right]$$

if $m=-1 \Rightarrow \rho^{m+1}=1$, $\cos(m+1)t=1$, $\sin(m+1)t = 0 \Rightarrow I = 2\pi i$

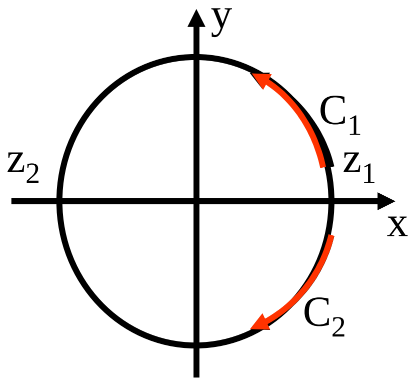
if $m\neq-1 \Rightarrow I = 0$

$$\therefore \quad \oint_C (z-z_o)^m\, dz = \begin{cases} 2\pi i & m = -1 \\ 0 & m \neq -1 \end{cases}$$

Example III

Example $\int_{z_1}^{z_2} (1/z)dz$, z_1, z_2 two points on the unit circle

e.g. $z_1 = (1,0)$, $z_2 = (-1,0)$



along C_1 $\int_{z_1}^{z_2} (1/z)dz = \pi i$

along C_2 $\int_{z_1}^{z_2} (1/z)dz = -\pi i$

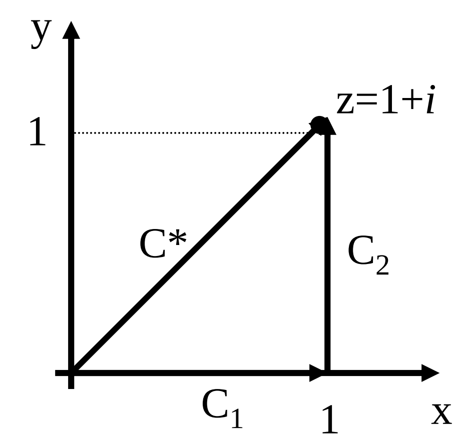
In general, a complex line integral depends not only on the end points of the path but also on the path itself.

Different Paths Different Values

Integrate $f(z) = \text{Re } z = x$ from 0 to $1+i$

(a) along C*; (b) along C consisting of C_1 and C_2

(a) C* can be represented by $z(t) = t+i\,t$ $\quad 0 \le t \le 1$



$dz(t)/dt = 1+i$ and $f[z(t)] = x(t) = t$

$$I^* = \int_{C^*} \text{Re}(z)dz = \int_0^1 t(1+i)dt = \boxed{(1+i)/2}$$

(b) C_1 can be represented by $z(t) = t$ $\quad (0 \le t \le 1)$

$dz(t)/dt = 1$ and $f[z(t)] = x(t) = t$

C_2 can be represented by $z(t) = 1+i\,t$ $\quad (0 \le t \le 1)$

$dz(t)/dt = i$ and $f[z(t)] = x(t) = 1$

$$I = \int_{C_1} \text{Re}(z)dz + \int_{C_2} \text{Re}(z)dz = \int_0^1 t\,dt + \int_0^1 1\,i dt = \boxed{(1/2)+i}$$

Upper Bound of Integral Value

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is the **ML** inequality

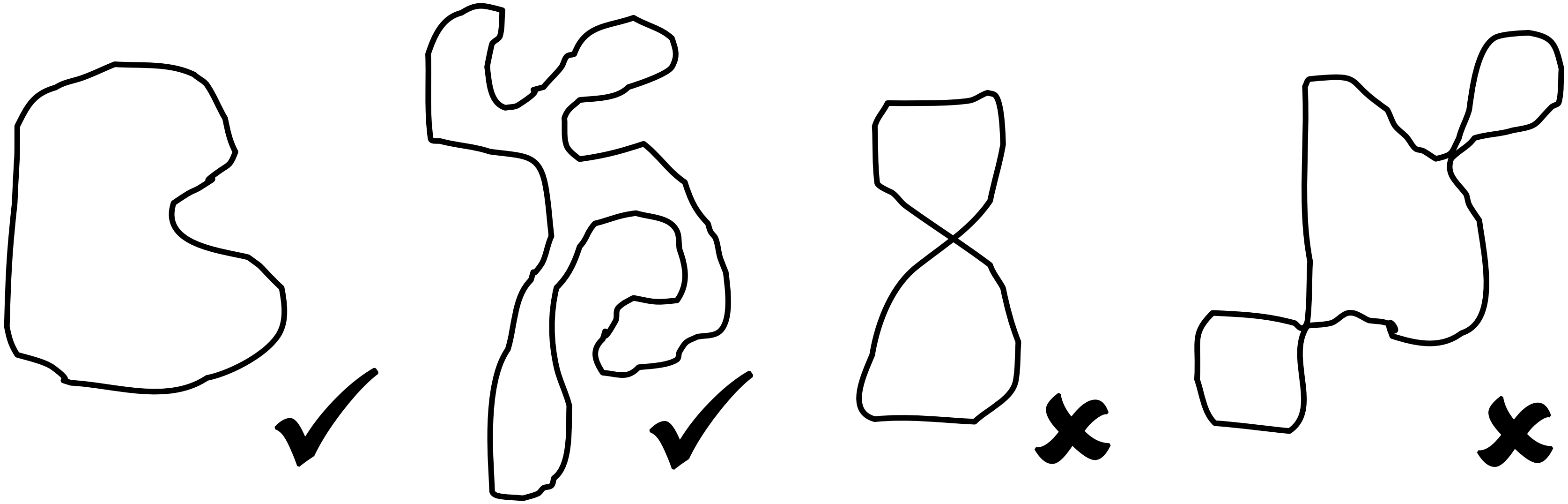
$$\left| \int_C f(z)dz \right| \leq \mathbf{ML}$$

where **L** is the length of C
and **M** is a constant such that $|f(z)| \leq \mathbf{M}$ everywhere on C

The proof is relatively straight forward. See book.

Simple Closed Path

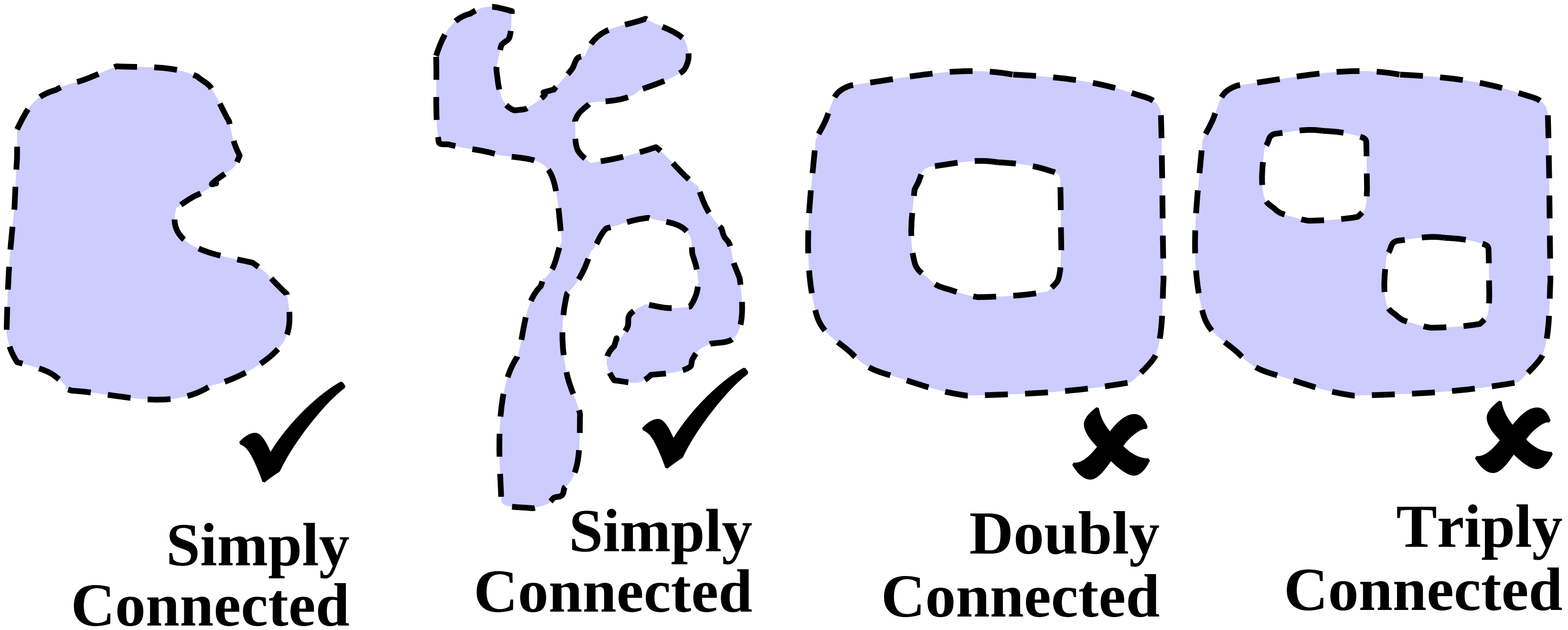
A **simple closed path** (a contour) is a closed path that does not intersect or touch itself



Simple Connected Domain

A **simple connected domain** D in the complex plane is a domain such that every simple closed path in D encloses only points of D

A domain that is not simply connected is called **multiply connected**.

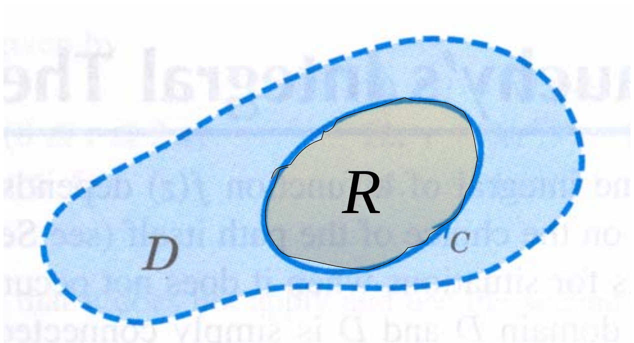


Green's Theorem in the Plane

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $u(x, y)$ and $v(x, y)$ be functions that are continuous and have continuous partial derivatives $\partial u / \partial y$ and $\partial v / \partial x$ everywhere in some domain containing R. Then

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = \oint_C \left(udx + vdy \right)$$

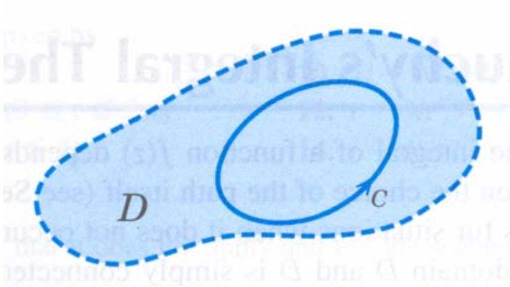
Here we integrate along the entire boundary C of R such that R is on the left as we advance in the direction of integration.



Cauchy’s Integral Theorem

If f(z) is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)dz = 0$$



If f(z) is continuous then

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy) = \oint_C udx - \oint_C vdy + i[\oint_C udy + \oint_C vdx]$$

if $f^{/}(z)$ is continuous then we can use Green’s Theorem:

$$\iint_R (\partial v/\partial x - \partial u/\partial y)dxdy = \oint_C (udx+vdy)$$

So that:

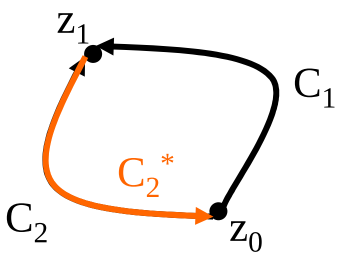
$$\oint_C f(z)dz = \iint_R (-\partial v/\partial x - \partial u/\partial y)dxdy + i\iint_R (\partial u/\partial x - \partial v/\partial y)dxdy$$

$$= 0 \text{ (by CREs)}$$ (the condition $f^{/}(z)$ can be removed see appendix 4 for proof)

Theorem (Independence of Path)

If f(z) is analytic in a simply connected domain D, then the integral f(z) is independent of path in D

Let $\quad I_1 = \int_{C_1} f(z)dz\ ,\quad I_2 = \int_{C_2} f(z)dz$



$$I_2^* = \int_{C_2^*} f(z)dz = -I_2$$

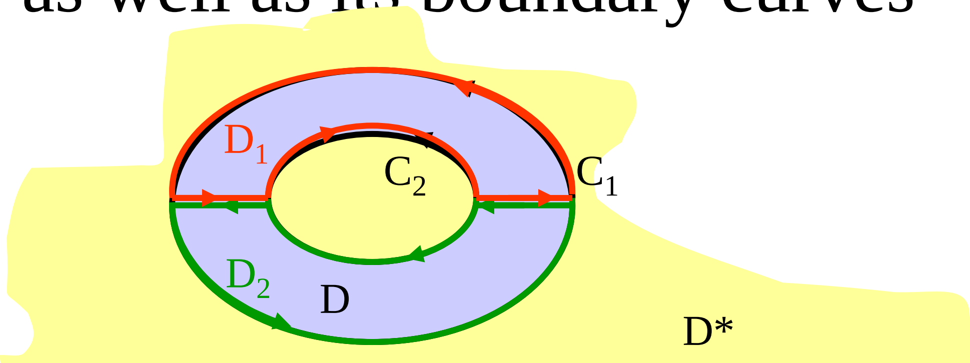
By Cauchy's integral theorem, $I_1+I_2^* = 0$ implies $I_1-I_2 = 0$

therefore $I_1=I_2$

Principle of deformation of path: For a given integral of an analytic function we may impose a continuous deformation on the path of integration (keeping the end points fixed) as long as the deforming path never contains any points at which f(z) is not analytic. The value of the line integral remains the same under the deformation.

Multiply Connected Domains

For a **doubly connected domain** D with outer boundary curve C_1 and inner C_2, if a function f(z) is analytic in any domain D* that contains D as well as its boundary curves then:
$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$



Apply Cauchy's theorem to D_1 and D_2, noting that the integrals over the horizontal lines in both directions cancel out themselves.

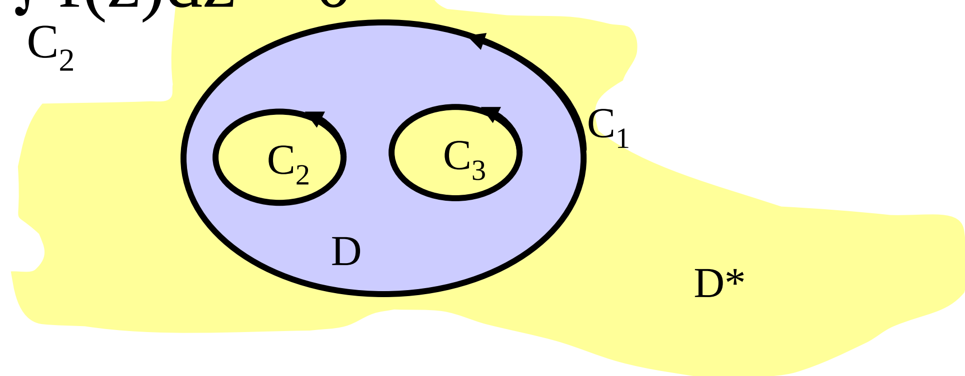
$$I_1 = I_{C1} + I_{L\text{-}R} + I_{C2} + I_{L\text{-}R}$$

$$I_2 = I_{C1} + I_{R\text{-}L} + I_{C2} + I_{R\text{-}L}$$

$$I_1 = I_2 = 0 \text{ and } I_1 + I_2 = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

Similarly, for triply connected

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$



Example

$$(*) \qquad \oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \neq -1) \end{cases}$$

for counterclockwise integration around any simple closed path containing z_0 in its interior

We have already shown that (*) is true when C is a circle of radius ρ with center z_0. By the above theorem for doubly connected domain (*) is true

Indefinite Integral

Theorem If f(z) is analytic in a simply connected domain D, then there exists an indefinite integral F(z) of f(z) in D ($F'(z) = f(z)$) which is analytic in D and for all paths in D joining any two points z_0 and z_1 in D, the integral of f(z) from z_0 to z_1 can be evaluated by

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

The proof is relatively straightforward and is easy to follow in Kreyszig.....

Examples

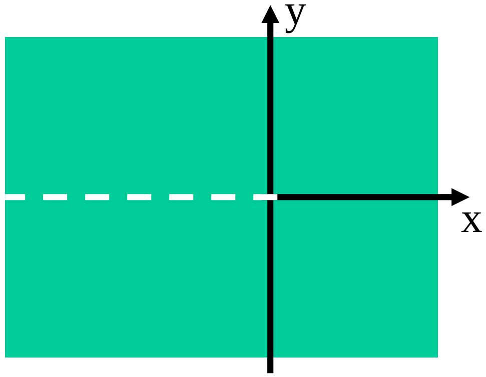
$$\int_0^{1+i} z^2 dz = (1/3)z^3\Big|_0^{1+i} = (1/3)(1+i)^3 = -2/3 + 2i/3$$

$$\int_{-\pi i}^{\pi i} \cos z \, dz = \sin z \,\Big|_{-\pi i}^{\pi i} = 2 \sin(\pi i) = 2i\sinh(\pi) = 23.097i$$

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2} \, dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2 \left(e^{4-3\pi i/2} - e^{4+\pi i/2}\right) = 0$$

$$\int_{-i}^{i} (1/z) \, dz = \mathrm{Ln}\, z\Big|_{-i}^{i} = i\pi/2 - (-i\pi/2) = i\pi$$

Here D is the complex plane without 0 and the negative real axis (a simply connected domain)



Examples II

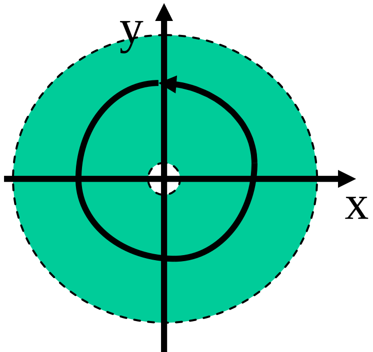
Simple connectedness is essential for integration using indefinite integral.

If $z_1=z_0$, $\int_{z_0}^{z_1} f(z)dz = F(z_0) - F(z_1) = 0$

i.e. the integral over a closed path is zero.... BUT

$\oint_C (1/z)\, dz = 2\pi i$ (counterclockwise over the unit circle)

This contradiction is due to the fact that 1/z is NOT analytic at z=0. Although 1/z is analytic in the annulus which doesn't include the origin this domain is not simply connected.



Cauchy's Integral Formula

Theorem: Let f(z) be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0

$$(1) \qquad \oint_C f(z)/(z-z_0)\,dz = 2\pi i\, f(z_0) \qquad \text{(Cauchy's integral formula)}$$

the integration being taken counter clockwise

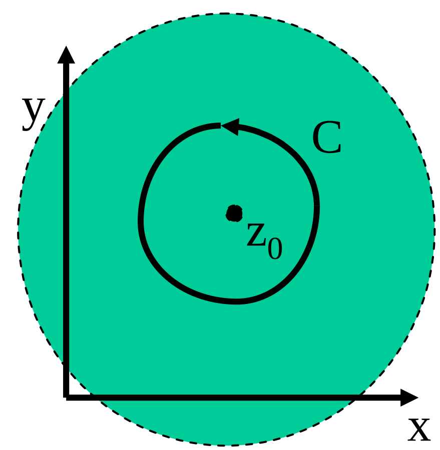
Proof By addition and subtraction $f(z) = f(z_0)+[f(z)-f(z_0)]$

$$I = \oint_C f(z)/(z-z_0)\,dz = f(z_0)\oint_C 1/(z-z_0)\,dz + \oint_C (f(z)-f(z_0))/(z-z_0)\,dz$$

$$= I_1 + I_2$$

where $I_1 = f(z_0)2\pi i$

we now need to show that $I_2 = 0$



Cauchy's Integral Formula

Since the integrand I_2 is analytic except at $z=z_0$, by the *principle of deformation of path* we can replace C by a small circle K of radius ρ and center z_0

Since f(z) is analytic, it is continuous. Therefore for any given $\varepsilon>0$ we can find a $\delta>0$ such that

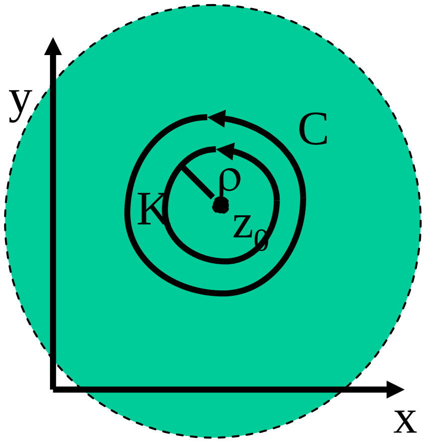
$$|f(z)-f(z_0)| < \varepsilon \qquad\qquad \text{if } |z-z_0| < \delta$$

Choosing the radius ρ of K smaller than δ, we have

$$|(f(z)-f(z_0))/(z-z_0)| < \varepsilon/\rho \quad \text{on K}$$

By the *ML inequality*

$$|I_2| = \left|\oint (f(z)-f(z0))/(z-z_0)\, dz\right| < (\varepsilon/\rho)2\pi\rho$$



As we make ε arbitrarily small → $I_2 \to 0$.

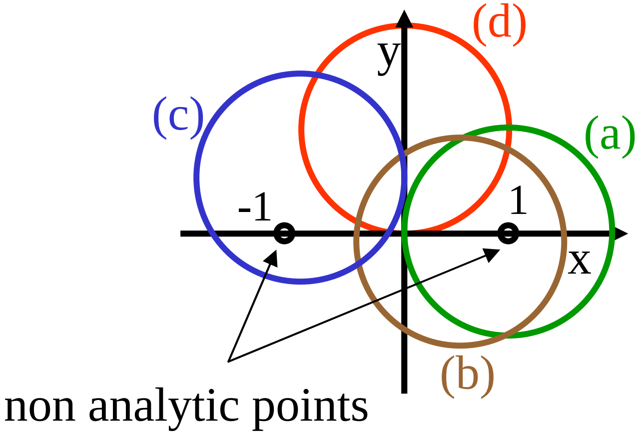
Examples

$$\oint_C e^z/(z-3)\,dz = \begin{cases} 2\pi i e^3 & \text{if C encloses } z_0=3 \\ 0 & \text{if } z_0=3 \text{ lies outside C} \end{cases}$$

If C encloses $z_0 = i/2$

$$\oint_C (z^3-6)/(2z-i)dz = \oint_C (\tfrac{1}{2}z^3-3)/(z-i/2)dz = 2\pi i[\tfrac{1}{2}z^3-3]\Big|_{z=i/2} = \pi/8-6\pi i$$

Integrate $g(z) = (z^2+1)/(z^2-1)$, i.e. find $I = \oint_C g(z)dz$ in the counterclockwise sense around a circle of radius 1 with center at (a) $z=1$ (b) $z=1/2$ (c) $z=-1+i/2$ (d) $z=i$



Examples

$$\oint_C e^z/(z-3)\,dz = \begin{cases} 2\pi i e^3 & \text{if C encloses } z_0=3 \\ 0 & \text{if } z_0=3 \text{ lies outside C} \end{cases}$$

If C encloses z0 = $i/2$

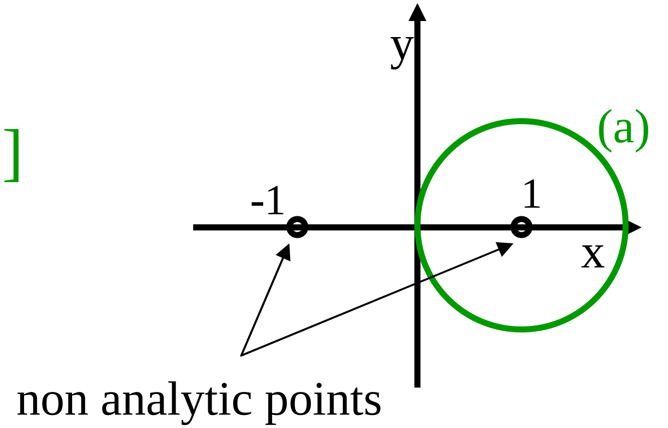
$$\oint_C (z^3-6)/(2z-i)dz = \oint_C (\tfrac{1}{2}z^3-3)/(z-i/2)dz = 2\pi i[\tfrac{1}{2}z^3-3]\Big|_{z=i/2} = \pi/8-6\pi i$$

Integrate $g(z) = (z^2+1)/(z^2-1)$, i.e. find $I = \oint_C g(z)dz$ in the counterclockwise sense around a circle of radius 1 with center at (a) $z=1$ (b) $z=1/2$ (c) $z=-1+i/2$ (d) $z=i$

(a) $z_0 = 1$, $z-z_0 = z-1$

$g(z) = (z^2+1)/(z^2-1) = [(z^2+1)/(z+1)][1/(z-1)]$

$\therefore$ $f(z) = (z^2+1)/(z+1)$ and $I = 2\pi i\, f(1) = 2\pi i$



Examples

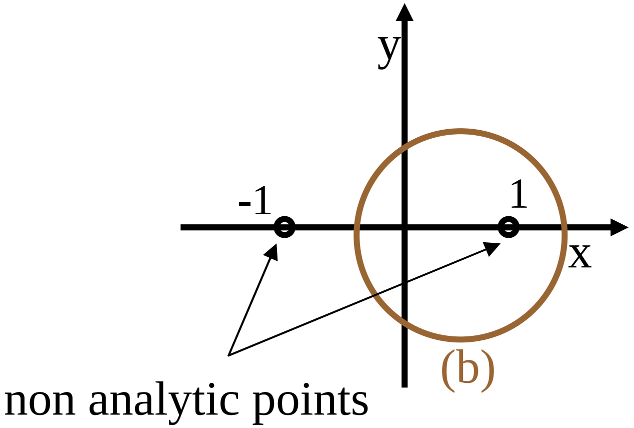
$$\oint_C e^z/(z-3)\,dz = \begin{cases} 2\pi i e^3 & \text{if C encloses } z_0=3 \\ 0 & \text{if } z_0=3 \text{ lies outside C} \end{cases}$$

If C encloses z0 = $i/2$

$$\oint_C (z^3-6)/(2z-i)dz = \oint_C (\tfrac{1}{2}z^3-3)/(z-i/2)dz = 2\pi i[\tfrac{1}{2}z^3-3]\Big|_{z=i/2} = \pi/8-6\pi i$$

Integrate $g(z) = (z^2+1)/(z^2-1)$, i.e. find $I = \oint_C g(z)dz$ in the counterclockwise sense around a circle of radius 1 with center at (a) $z=1$ (b) $z=1/2$ (c) $z=-1+i/2$ (d) $z=i$

(b) is the same as (a) = $2\pi i$



Examples

$$\oint_C e^z/(z-3)\,dz = \begin{cases} 2\pi i e^3 & \text{if C encloses } z_0=3 \\ 0 & \text{if } z_0=3 \text{ lies outside C} \end{cases}$$

If C encloses z0 = $i/2$

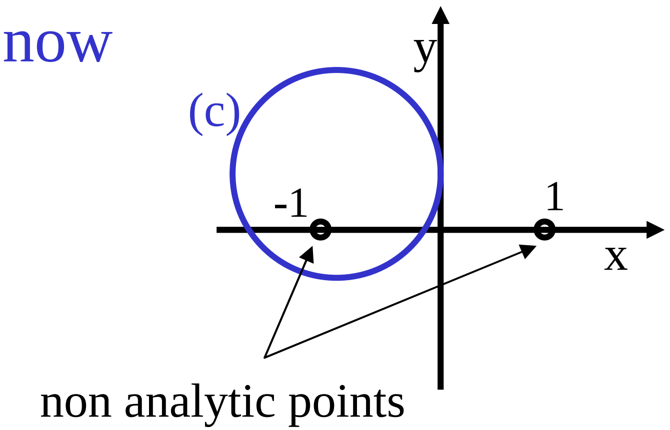
$$\oint_C (z^3-6)/(2z-i)dz = \oint_C (\tfrac{1}{2}z^3-3)/(z-i/2)dz = 2\pi i[\tfrac{1}{2}z^3-3]\Big|_{z=i/2} = \pi/8-6\pi i$$

Integrate $g(z) = (z^2+1)/(z^2-1)$, i.e. find $I = \oint_C g(z)dz$ in the counterclockwise sense around a circle of radius 1 with center at (a) $z=1$ (b) $z=1/2$ (c) $z=-1+i/2$ (d) $z=i$

(c) g(z) is the same but f(z) changes as $z_0 = -1$ now

$z-z_0 = z+1$, $f(z) = (z^2+1)/(z-1)$

$I = 2\pi i\, f(-1) = 2\pi i\, (z2+1)/(z-1) = -2\pi i$



Examples

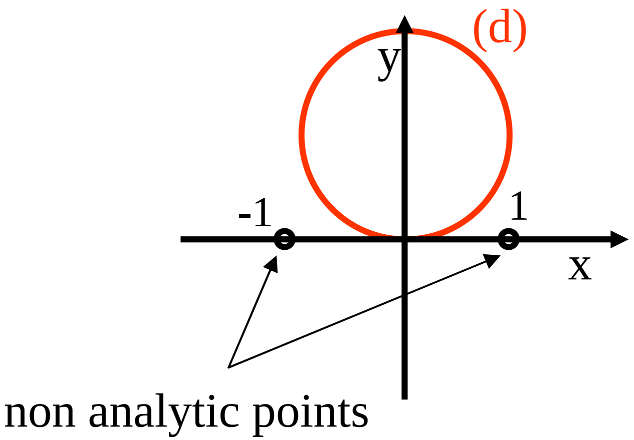
$$\oint_C e^z/(z-3)\,dz = \begin{cases} 2\pi i e^3 & \text{if C encloses } z_0=3 \\ 0 & \text{if } z_0=3 \text{ lies outside C} \end{cases}$$

If C encloses z0 = $i/2$

$$\oint_C (z^3-6)/(2z-i)dz = \oint_C (\tfrac{1}{2}z^3-3)/(z-i/2)dz = 2\pi i[\tfrac{1}{2}z^3-3]\Big|_{z=i/2} = \pi/8-6\pi i$$

Integrate $g(z) = (z^2+1)/(z^2-1)$, i.e. find $I = \oint_C g(z)dz$ in the counterclockwise sense around a circle of radius 1 with center at (a) $z=1$ (b) $z=1/2$ (c) $z=-1+i/2$ (d) $z=i$

(d) Clearly I=0 by Cauchy's Integral formula



Example II

Calculate $I=\oint_C \tan z/(z^2-1)\,dz$ C is the circle $|z|=3/2$

tan z is not analytic at $\pm\pi/2$, $\pm 3\pi/2$,..... but all these points lie outside the contour

$1/(z^2-1) = 1/[(z+1)(z-1)]$ is not analytic at +1 and -1

Note that

$$1/(z^2-1) = ½[1/(z-1) – 1/(z+1)]$$

$$I = ½[\oint_C \tan z/(z-1)\,dz - \oint_C \tan z/(z+1)\,dz]$$

$$=2\pi i/2[\tan(1) – \tan(-1)] = 2\pi i \tan(1) \approx 9.785i$$