ERG 2012B

Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #4:

More Functions & Complex Integration

Trigonometric Functions

By Euler Formula

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$\Rightarrow \quad \cos x = (e^{ix} + e^{-ix})/2$$

$$\sin x = (e^{ix} - e^{-ix})/2i$$

For complex values $z = x + i\,y$ it can be generalized to:

$$\cos z = (e^{iz} + e^{-iz})/2$$

$$\sin z = (e^{iz} - e^{-iz})/2i$$

Then we can define:

$\tan z = \sin z / \cos z$; $\quad \cot z = \cos z / \sin z$

$\sec z = 1/\cos z$; $\quad \csc z = 1/\sin z$

As e^z is entire, cos z and sin z are also entire functions

tan z, and sec z (cot z and csec z) are **not** entire as they are analytic except at the points where cos (or sin) is zero

Hyperbolic Functions

It is straightforward to show that

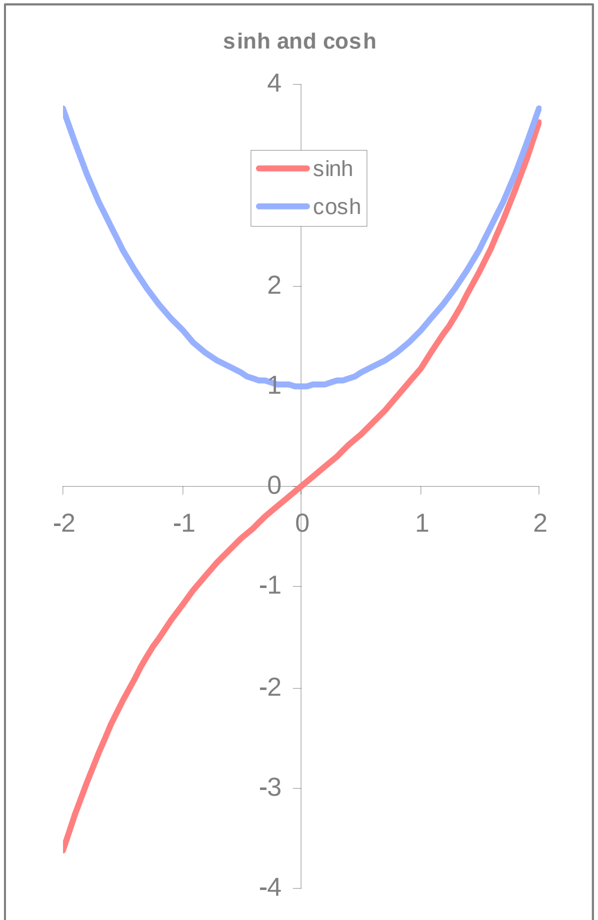
$$(\cos z)' = -\sin z; \quad (\sin z)' = \cos z; \ (\tan z)' = \sec^2 z$$

- The complex **hyperbolic cosine** and **sine** are defined by $\cosh z = ½(e^z+e^{-z})$; $\sinh z = ½(e^z-e^{-z})$

- These functions are entire and have derivatives $(\cosh z)' = \sinh z$; $(\sinh z)' = \cosh z$

- Other hyperbolic functions are defined by: $\tanh z = \sinh z / \cosh z$; $\coth z = \cosh z / \sinh z$; $\text{sech} = 1/\cosh z$; $\text{csech}\ z = 1/\sinh z$

- They are analytic except at the points where the denominator is zero



Hyperbolic Functions

- Complex trigonometric & hyperbolic functions are related

  cosh iz = cos z; sinh iz = i sin z

  cos iz = cosh z; sin iz = i sinh z

  This is simple to verify, giving the formulas for complex trigonometric functions:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

  and the formulas for complex hyperbolic functions:

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

Example

Example: Show that

a) $\cos z = \cos x \cosh y - i \sin x \sinh y$; $|\cos z|^2 = \cos^2 x + \sinh^2 y$

b) $\sin z = \sin x \cosh y + i \cos x \sinh y$; $|\sin z|^2 = \sin^2 x + \sinh^2 y$

Solution:

$$\cos z = [e^{i(x+iy)} + e^{-i(x+iy)}]/2$$

$$= e^{-y}(\cos x + i \sin x)/2 + e^{y}(\cos x - i \sin x)/2$$

$$= \tfrac{1}{2}(e^{y}+e^{-y})\cos x - \tfrac{1}{2} i (e^{y}-e^{-y})\sin x$$

$$\boxed{= \cosh y \cos x - i \sinh y \sin x}$$

$$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$

As $\cosh^2 y - \sinh^2 y = [\tfrac{1}{2}(e^{y}+e^{-y})]^2 - [\tfrac{1}{2}(e^{y}-e^{-y})]^2 = 1$

$$|\cos z|^2 = \cos^2 x (1+\sinh^2 y) + \sin^2 x \sinh^2 y = \mathbf{\cos^2 x + \sinh^2 y}$$

(b) can be verified in a similar fashion.

Example II

Example: Solve $\cos z = 5$

Note: cos z and sin z are still periodic as with real numbers BUT they are no longer bounded....

Solution:

$$\cos z = [e^{iz}+e^{-iz}]/2 = 5$$

$$e^{iz}+ e^{-iz} - 10 = 0$$

$$e^{i2z} - 10\, e^{iz} + 1 = 0$$

$$\boxed{\begin{array}{l} e^{iz} = e^{-y+ix} = e^{-y}(\cos x + i \sin x) \\ e^{iz} \text{ is real} \\ \quad \Rightarrow i \sin x = 0 \Rightarrow \cos x = 1 \end{array}}$$

$$e^{iz} = 5 \pm \sqrt{(25-1)} = 9.899 \text{ or } 0.101$$

$$\Rightarrow \begin{array}{l} e^{-y} = 9.899 \text{ or } 0.101 \qquad \Rightarrow y = \pm 2.292 \\ e^{ix} = 1 \Rightarrow x = \pm 2n\pi \quad (n=0,1,2....) \end{array}$$

$$\therefore z = \pm 2n\pi \pm 2.292\, i \quad (n=0,1,2....)$$

Example III

Example: Solve $\cos z = 0$

Solution:

$$\cos z = \cos x \cosh y - i \sin x \sinh y = 0$$

Real Part $= 0 \Rightarrow$ $\cos x = 0$; $x = \pm(2n+1)\pi/2$ $(n=0,1,2...)$

Imaginary Part $= 0 \Rightarrow \sinh y = 0$; $y = 0$

$$\therefore\ z = \pm(2n+1)\pi/2 \quad (n=0,1,2....)$$

Example: Solve $\sin z = 0$

Solution:

$$\sin z = \sin x \cosh y + i \cos x \sinh y = 0$$

Real Part $= 0 \Rightarrow$ $\sin x = 0$; $x = \pm n\pi$ $(n=0,1,2...)$

Imaginary Part $= 0 \Rightarrow \sinh y = 0$; $y = 0$

$$\therefore\ z = \pm n\pi \quad (n=0,1,2....)$$

Hence: Zeros of cos z and sin z are those of the real cos & sin

Logarithms

The **natural logarithm** of $z = x + i\,y$ is denoted by $\ln z$ and is defined as the inverse of the exponential function. i.e.

$w = \ln z$ is defined for $z \neq 0$ by the relation

$e^w = z$ ($z = 0$ is impossible since $e^w \neq 0$)

If we let $w = \ln z = u + i\,v$ and $z = r\,e^{i\theta}$ then

$$e^w = e^{u+iv} = e^u e^{iv} = re^{i\theta}$$

$$\Rightarrow \quad e^u = r \quad \Rightarrow \quad u = \ln r = \ln|z|$$

and $v = \theta$

$$\boxed{\Rightarrow \ \ln z = \ln|z| + i\theta \qquad (|z| > 0,\ \theta = \arg z)}$$

- As arg z is multi-valued ln z is also many valued
- $\mathrm{Ln}\, z = \ln|z| + i\,\mathrm{Arg}\, z$ $(z \neq 0)$ - Ln z is single valued
- $\ln z = \mathrm{Ln}\, z \pm 2n\pi i$ (n-0,1,2.........)

Example

$\ln(1) = 0, \pm 2\pi i, \pm 4\pi i, \ldots\ldots$ $\quad$ $\mathrm{Ln}(1) = 0$

$\ln(-1) = \pm\pi i, \pm 3\pi i, \pm 5\pi i\ldots\ldots$ $\quad$ $\mathrm{Ln}(-1) = \pi i$

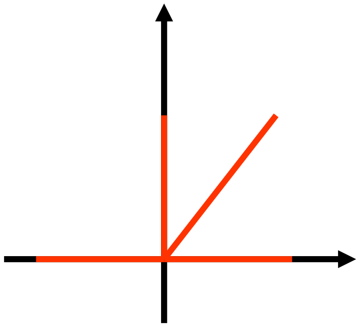
$\ln(i) = \pi i/2, -3\pi i/2, +5\pi i/2,\ldots\ldots$ $\quad$ $\mathrm{Ln}(i) = \pi i/2$

$\ln(1+i) = \ln\sqrt{2} + (\pi/4 \pm 2n\pi)i$ (n=0,1,2,...) $\quad$ $\mathrm{Ln}(1+i)=\ln\sqrt{2}+\pi i/4$

- The familiar relations for the natural log continue to hold for complex values i.e.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

$$\ln(z_1/z_2) = \ln z_1 - \ln z_2$$



BUT the relations are to be understood in the sense that each value of one side is also contained among the values of the other side.

Example II

Let $z_1 = z_2 = e^{\pi i} = -1$

then $\ln(z_1 z_2) = \ln(z_1)+\ln(z_2) = \pi i+\pi i = 2\pi i = \ln(1)$

But it is not true for the principal values.

$\mathrm{Ln}\, z = \ln|z| + i\, \mathrm{Arg}\, z \Rightarrow \mathrm{Ln}\, z_1 = \mathrm{Ln}\, z_2 = \pi i$,

and $\mathrm{Ln}\,(z_1 z_2) = \mathrm{Ln}(1) = 0$ ($\neq \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2$).

- $\ln(e^z) = \ln(e^{x+iy}) = \ln(e^x) + i\, y \pm 2n\pi i$

  $= z \pm 2n\pi i, \quad n = 0, 1, 2, \ldots$

compare with the real case, where $\ln(e^x) = x$.

Differential of ln z

- For each non-negative integer *n* the expression

  $$\ln z = \mathrm{Ln}\, z \pm 2n\pi i$$

  defines a function.
- Each of these functions is analytic except at z=0 and on the negative real axis *(where even the imaginary part is not continuous but jumps by 2π)*
- We can show this by proving that $(\ln z)^{/} = 1/z$

Let $\ln z = u + i\, v$.

then $u = \ln|z| = ½\ln(x^2+y^2)$; $v = \arg z = \tan^{-1} y/x + c$

$$u_x = x/(x^2+y^2) = v_y = [1/(1+(y/x)^2)](1/x)$$

$$u_y = y/(x^2+y^2) = -v_x = -[1/(1+(y/x)^2)](-y/x)$$

CREs satisfied

$$(\ln z)^{/} = u_x + i\, v_x \;(= -i u_y + v_y)$$

$$= x/(x^2+y^2) - i\, y/(x^2+y^2)$$

$$= (x - i\, y)/(x^2+y^2) = 1/z$$

General Powers

- General powers of a complex number $z = x + i\,y$ defined:

  $$z^c = e^{c \ln z} \qquad \text{(c complex, } z \neq 0\text{)}$$

- Since ln(z) is multi-valued z^c will also be multi-valued
- The **principal** value of $z^c = e^{c\,\mathrm{Ln}(z)}$
- If $c = n = 1,2,\ldots.$ then z^n is single valued and identical to the usual n^{th} power of z
- If $c = n = -1,-2,\ldots.$ the situation is similar
- If $c = 1/n = 2,3,\ldots.$ then

  $$z^c = \sqrt[n]{z} = e^{(1/n)\ln\, z} \qquad (z \neq 0)$$

  the exponent is determined up to multiples of $2\pi i/n$ and we obtain n distinct values of the n^{th} root
- If $c = p/q$, the quotient of two positive integers then z^c has a finite number of distinct values
- If c is real irrational or complex, then z^c is infinitely many valued

Example

$i^i = e^{i \ln i} = \exp[i(\pi i/2 \pm 2n\pi i)] = e^{-\pi/2 \pm 2n\pi}$ (Note: real!)

the principal value (n=0) is $e^{-\pi/2}$

$(1+i)^{2-i} = \exp[(2 - i)\ln(1 + i)]$

$= \exp[(2-i)\{\ln(\sqrt{2}) + \pi i/4 \pm 2n\pi i\}]$

$= \exp[\ 2\ln(\sqrt{2}) + \pi i/2 \pm 4n\pi i - i\ln(\sqrt{2}) + \pi/4 \pm 2n\pi]$

$= 2e^{\pi/4 \pm 2n\pi}[\cos(\pi/2 \pm 4n\pi - \ln(\sqrt{2})) + i\sin(\pi/2 \pm 4n\pi - \ln(\sqrt{2}))]$

$= 2e^{\pi/4 \pm 2n\pi}[\sin(½\ln(2)) + i\cos(½\ln(2))]$

- It is conventional for real positive z = x, z^c means $e^{c \ln(x)}$, where ln(x) is the *elementary real natural logarithm*

- If z=e then $z^c = e^c$ (with c=a+b*i*) yields a unique value:

  $e^c = e^a(\cos b + i \sin b)$

- For any complex number a, $a^z = e^{z \ln(a)}$