ERG 2012B
Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #3: Complex and Analytic Functions

Sets in the Complex Plane

The **complement** of a set S is the set of all points that do *not* belong to S.

A set S is called **closed** if its complement is open. *E.g. the points on and inside the unit circle form a closed set since its complement* $|z|>1$ *is open.*

A **boundary point** of a set S is a point every neighbour-hood of which contains both points that belong to S and points that don't. E.g. the boundary points of an annulus are the points on the two bounding circles.

If S is open, then ***no*** *boundary point belongs to S*

If S is closed then ***every*** *boundary point belongs to S*

A **region** is a set made up of a domain plus, perhaps, some or all of its boundary points *(warning: some authors use region to mean domain)*

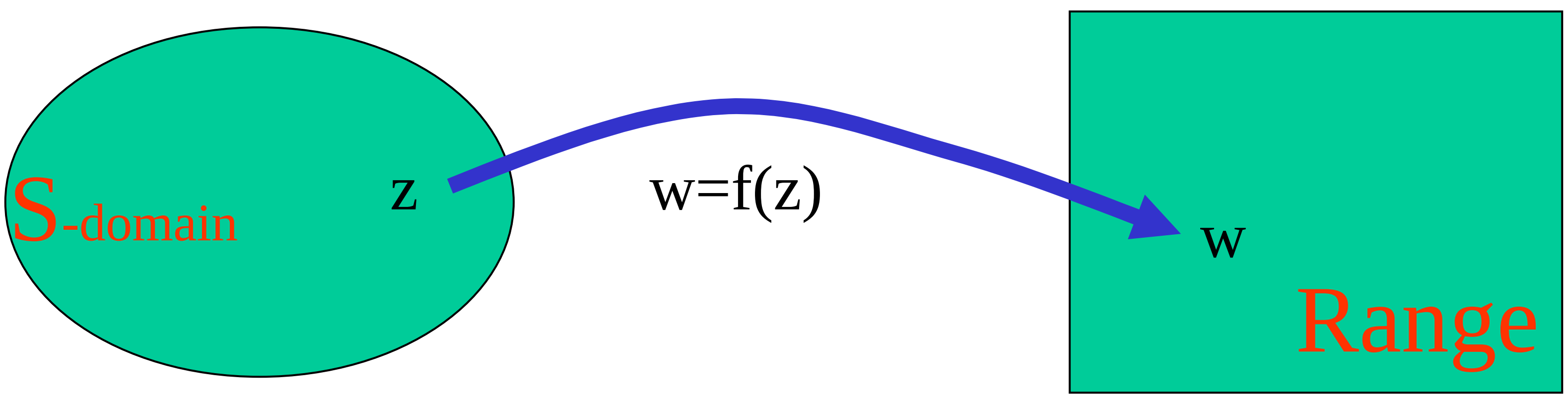
Complex Functions

Let "S" be a set of complex numbers "z"

- A **function** "f" defined on S is a rule that assigns to every z in S a complex number "w" called the **value** of f at z

$$w = f(z)$$

  - z is a **complex variable**
  - S is the definition **domain** of f
  - the set of all values of a function f is called the **range** of f



Complex Functions

As w is complex ($w = u + i\,v$; u and v are real) and $z = x + i\,y$, we can write

$$w = f(z) = u(z) + i\,v(z) = u(x,y) + i\,v(x,y)$$

– A ***complex*** function f(z) is equivalent to a pair of ***real*** functions u(x,y) and v(x,y), each depending on two ***real*** variables x and y.

Examples

Example 1

If $w = f(z) = z^2 + 3z$.

Find u and v and $f(1 + 3i)$

Solution $z = x + i\,y$, $\quad (i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1)$

$$f(z) = (x + i\,y)^2 + 3(x + i\,y) = x^2 + i2xy - y^2 + 3x + i3y$$

$$= x^2 - y^2 + 3x + i\,(2xy + 3y) = u(x,y) + i\,v(x,y)$$

$$\Rightarrow \quad u = \mathrm{Re}(f(z)) = x^2 - y^2 + 3x$$

$$v = \mathrm{Im}(f(z)) = 2xy + 3y$$

$$f(1+3i) = (1+3i)^2 + 3(1+3i) = 1-9+3+6i+9i = -5+15i$$

or $\quad f(1+3i) = 1-9+3+i(6+9) = -5+15i$

Examples

Example 2

If $w = f(z) = 2iz + 6\bar{z}$

Find u and v and $f(1/2 + 4i)$

Solution

$$
\begin{aligned}
f(z) &= 2i(x + i\,y) + 6(x - i\,y) \\
&= 2ix - 2y + 6x - 6iy \qquad (i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ u &= \mathrm{Re}(f(z)) = 6x-2y \\
v &= \mathrm{Im}(f(z)) = 2x-6y
\end{aligned}
$$

$$f(1/2+4i) = 3 - 8 + i(1-24) = -5 - 23i$$

Limits

Definition

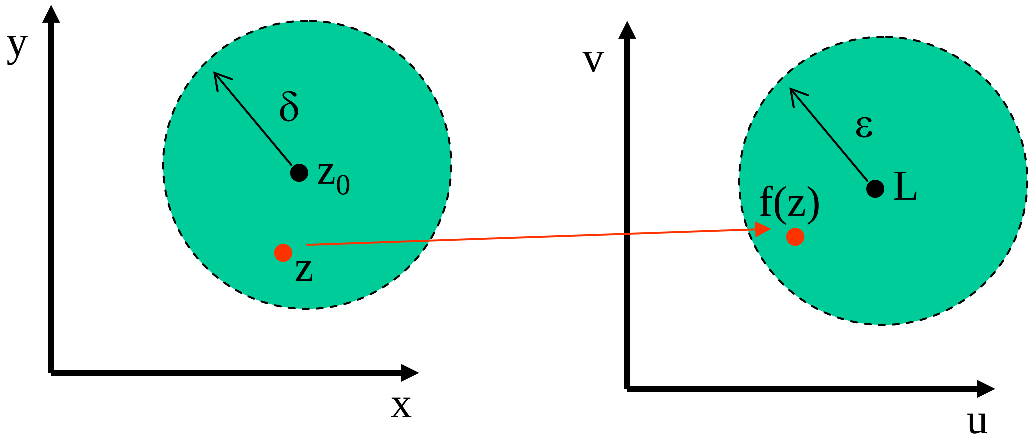
A function "f(z)" is said to have the limit "L" as "z" approaches a point "z_0", written

$$\lim_{z \to z_0} f(z) = L$$

if "f" is defined in a neighbourhood of z_0 (except maybe at z_0 itself) and if

$\forall$ real $\varepsilon > 0$, $\exists$ a real $\delta > 0$ s.t.

$\forall\, z \neq z_0$, and $|z-z_0| < \delta$, then $|f(z)-L| < \varepsilon$



If a limit exists, it is unique.

Continuous Functions

Definition

A function "f(z)" is said to be **continuous** at $z=z_0$ if

$f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Definition

f(z) is said to be **continuous in a domain** if it is continuous at each point of this domain

Definition

The **derivative** of a complex function at a point z_0 $f'(z_0)$ is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z-z_0}$$

provided the limit exists.

Then f is said to be **differentiable** at z_0

Example

Example

Show that $f(z) = z^2$ is differentiable for all z and $f'(z)=2z$

Solution

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z+\Delta z) = 2z$$

The differentiation rules are the same as in calculus of real numbers: Let c = constant, f, g are functions, then

$(c\, f)' = c\, f'$

$(f+g)' = f' + g'$

$(f\, g)' = f'\, g + f\, g'$

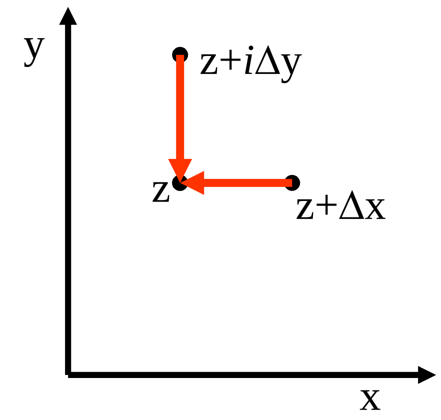
$(f/g)' = (f'\, g - f\, g')/g^2$

The chain rule also holds.

Example

Example

Show that $f(z) = \bar{z}$ is not differentiable



Solution

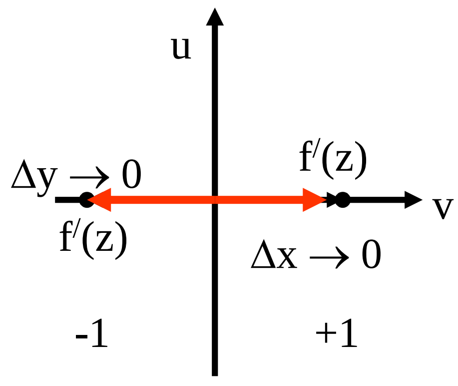
$f(z) = x - i\,y$

Let $\Delta z = \Delta x + i\,\Delta y$

$[f(z+\Delta z) - f(z)]/\Delta z = [(\overline{z+\Delta z}) - \bar{z}]/\Delta z = \overline{\Delta z} / \Delta z$

$$= \frac{(\Delta x - i\Delta y)}{(\Delta x + i\Delta y)} \rightarrow \begin{cases} 1 \ \text{ if } \Delta y = 0 & \text{(horizontal)} \\ -1 \ \text{ if } \Delta x = 0 & \text{(vertical)} \end{cases}$$

By definition, $f'(z)$ does not exist at any z



Analytic Functions

Definition

A function f(z) is said to be **analytic in a domain D** if f(z) is defined and differentiable at all points of D

The function f(z) is said to be **analytic at a point $z=z_0$** in D if f(z) is analytic in a neighbourhood of z_0

Definition

An **analytic function** is a function that is analytic in some domain.

Example - Polynomials

$$f(z) = c_0+c_1z+c_2z^2+.....+c_nz^n$$

where c_0, $c_1 ... c_n$ are complex constants.

f is analytic in the entire complex plane.

Analytic Functions - Examples

Rational Functions

$f(z) = g(z)/h(z)$

where $g(z)$ and $h(z)$ are two polynomials that have no common factors

f is analytic except, perhaps, at the points where $h(z) = 0$

Partial Fractions

$f(z) = c/(z-z_0)^m$

where c and z_0 are complex and m is a positive integer.

f is analytic except at z_0

Cauchy-Riemann Equations

The Cauchy-Riemann equations are among the most important equations in complex analysis and are one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$f(z) = u(x,y) + i\, v(x,y).$$

Roughly speaking, f is analytic in a domain D if and only if u and v have **continuous** first partial derivatives that satisfy the **Cauchy-Reimann Equations:**

$$u_x = v_y$$
$$u_y = - v_x \qquad \text{(CREs)}$$

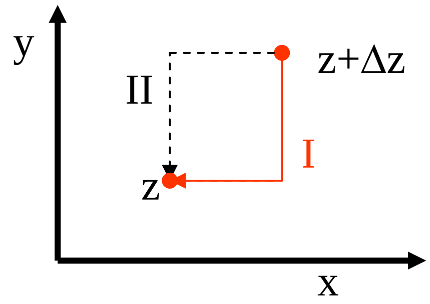
everywhere in D. Here

$$u_x = \frac{\partial u}{\partial x} \text{ and } u_y = \frac{\partial u}{\partial y} \qquad v_x = \frac{\partial v}{\partial x} \text{ and } v_y = \frac{\partial v}{\partial y}$$

Necessary and Sufficient Conditions

Relatively simple to show that the CREs are Necessary conditions for analyticity

We simply evaluate $f'(z)$ taking the limit $\Delta z \rightarrow 0$ along the two paths I and II



path I: let $\Delta y \rightarrow 0$ first and then $\Delta x \rightarrow 0$

path II: let $\Delta x \rightarrow 0$ first and then $\Delta y \rightarrow 0$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{[u(x+\Delta x, y+\Delta y)+i\, v(x+\Delta x, y+\Delta y)]-[u(x,y)+i\, v(x,y)]}{\Delta x + i\, \Delta y}$$

path I: (let $\Delta y \rightarrow 0$ first)

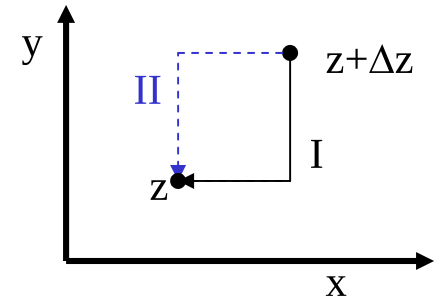
$$f'(z) = \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\boxed{f'(z) = u_x + i\, v_x}$$

Necessary and Sufficient Conditions

Relatively simple to show that the CREs are Necessary conditions for analyticity

We simply evaluate $f^{/}(z)$ taking the limit $\Delta z \rightarrow 0$ along the two paths I and II



path I: let $\Delta y \rightarrow 0$ first and then $\Delta x \rightarrow 0$

path II: let $\Delta x \rightarrow 0$ first and then $\Delta y \rightarrow 0$

$$f^{/}(z) = \lim_{\Delta z \rightarrow 0} \frac{[u(x+\Delta x, y+\Delta y)+i\, v(x+\Delta x, y+\Delta y)]-[u(x,y)+i\, v(x,y)]}{\Delta x + i\, \Delta y}$$

path II: (let $\Delta x \rightarrow 0$ first)

$$f^{/}(z) = \lim_{\Delta y \rightarrow 0} \frac{u(x,y+\Delta y) - u(x,y)}{i\, \Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x,y+\Delta y) - v(x,y)}{i\, \Delta y}$$

$\boxed{f^{/}(z) = u_x + i\, v_x}$ $\boxed{f^{/}(z) = -i u_y + v_y}$ Comparison give CREs

Necessary and Sufficient Conditions

Relatively simple to show that the CREs are Necessary conditions for analyticity

BUT, more difficult to show that they are **sufficient** conditions for analyticity.

(See appendix 4 in Kreyszig for details of this)

CREs Examples I

$f(z) = z^2 = x^2 - y^2 + i\,2xy$ is analytic $\forall$ z

$u = x^2 - y^2$

$v = 2xy$

clearly $u_x = 2x,\ v_y = 2x$

$u_y = -2y,\ v_x = 2y$

So CREs are satisfied, hence function is analytic....

BUT

$f(z) = \bar{z} = x - i\,y$

$u = x,\quad v = -y$

$u_x = 1 \neq v_y = -1$ so f(z) is **not** analytic

CREs Examples II

Is $f(z) = z^3$ analytic?

$$f(z) = (x + i\,y)^3 = x^3 + i\,3x^2y - 3y^2x - i\,y^3$$

$$u = x^3 - 3xy^2 \quad , \quad v = 3x^2y - y^3$$

and $\qquad u_x = 3x^2 - 3y^2,\ v_y = 3x^2 - 3y^2$

$$u_y = -6xy,\ v_x = 6xy$$

So CREs are satisfied, hence function is analytic.

Now find the most general analytic function f(z) which has a real part $u = x^2 - y^2 - x$

from CREs $\qquad u_x = 2x - 1 = v_y$

$$\Rightarrow \quad v = \int v_y dy + k(x) = 2xy - y + k(x)$$

and $\qquad u_y = -2y = -v_x = -2y - dk(x)/dx$

$$\Rightarrow \quad dk/dx = 0 \quad \Rightarrow k = \text{const (real)}$$

$$\Rightarrow \quad f(z) = u + i\,v = x^2 - y^2 - x + i(2xy - y + k) = z^2 - z + ik$$

CREs Examples III

Show that if $f(z)$ is analytic in D and $|f(z)| = k = \text{const}$ in D then $f(z) = \text{const}$ in D

Solution

$|f(z)| = k \Rightarrow u^2 + v^2 = k^2$

$\Rightarrow 2u\, u_x + 2v\, v_x = 0\ ,\ 2u\, u_y + 2v\, v_y = 0$

by CREs

(a) $u\, u_x - v\, u_y = 0$, (b) $u\, u_y + v\, u_x = 0$

u.(a).... $u^2 u_x - u\, v\, u_y = 0$

v.(b).... $u\, v\, u_y + v^2 u_x = 0$

(+)...... $(u^2 + v^2)\, u_x = 0$

u.(b)-v.(a).. $(u^2+v^2)u_y = 0$

if $k^2=u^2+v^2 = 0$ then $u=v=0 \Rightarrow f=0$ (const)

if $k \neq 0$ then $u_x=u_y=v_x=v_y=0 \Rightarrow u=\text{const}, v=\text{const} \Rightarrow f=\text{const}$

CREs in Polar Form

Let $z = r(\cos\theta + i\sin\theta)$

$f(z) = u(r,\theta) + i\, v(r,\theta)$

CREs now take the following form:

$$u_r = v_\theta/r\,, \quad v_r = -u_\theta/r \qquad (r > 0)$$

or $$ru_r = v_\theta, \quad rv_r = -u_\theta \qquad (r > 0)$$

[You can satisfy yourself that this is true by repeating the manipulations performed to show the CREs in conventional form. Try it as an exercise for you!]

Laplace's Equation

Complex analysis is **very** important in engineering as both the real and imaginary parts of an analytical function satisfy Laplace's equation – the most important equation in physics. (occurs in gravitation, electrostatics, fluid flow, heat conduction......

Theorem: If f(z)=u(x,y)+*i* v(x,y) is analytic in domain D then u and v satisfy Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

and

$$\nabla^2 v = v_{xx} + v_{yy} = 0$$

in D and have continuous second partial derivatives in D

Laplace's Equation

Theorem: If f(z)=u(x,y)+*i* v(x,y) is analytic in domain D then u and v satisfy Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{and} \quad \nabla^2 v = v_{xx} + v_{yy} = 0$$

in D and have continuous second partial derivatives in D

From CREs

$$u_x = v_y \qquad u_y = -v_x$$

$$\Rightarrow \quad u_{xx} = v_{yx} \qquad u_{yy} = -v_{xy}$$

It can be shown that the derivative of an analytic function is also analytic $\Rightarrow$ u & v have continuous partial derivatives of all orders and $v_{xy} = v_{yx}$

$\Rightarrow v_{yx} = v_{xy} = -u_{yy} \Rightarrow u_{xx} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0$, etc...

Solutions having **continuous** second order partial derivatives are called **harmonic**

The theory of harmonic functions is **potential theory**

Example

If two harmonic functions u & v satisfy the CREs in a domain D and they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **conjugate harmonic function** of u in D

Example

Verify that $u = x^2-y^2-y$ is harmonic in the whole complex plane and find a conjugate harmonic function v of u

Solution

$u_x = 2x; \quad u_y = -2y-1$

$u_{xx} + u_{yy} = 2 - 2 = 0 \qquad \therefore$ u is harmonic

For v to be conjugate harmonic function of u we have (CREs)

$v_y = u_x = 2x; \quad v_x = -u_y = 2y+1$

$v = \int v_y dy = 2xy+h(x) \Rightarrow v_x = 2y+dh/dx \Rightarrow dh/dx = 1$

$\Rightarrow h(x) = x+c \Rightarrow v = 2xy + x + c$

The analytic function is therefore:

$f(z) = u + i\, v = x^2 - y^2 - y + i\,(2xy + x + c) = z^2 + i\,(z + c)$

Exponential Function

Consider the function:

$f(\theta) = \cos\theta + i \sin\theta$

where θ can take any value. Its derivative with respect to θ is

$df(\theta)/d\theta = -\sin\theta + i \cos\theta = i (\cos\theta + i \sin\theta) = i\, f(\theta)$

So the derivative is "proportional" to itself (times a constant "*i*").

and

$$\mathbf{e^{i\theta} = \cos\theta + i \sin\theta}$$ **- Euler's Formula**

Or any complex number can be written:

$$\mathbf{z = r(\cos\theta + i \sin\theta) = re^{i\theta}}$$

Exponential Function - Properties

$\exp(z) = e^z = e^{x+i\,y} = e^x e^{i\,y} = e^x (\cos y + i \sin y)$

Properties

- e^z is an **entire function** i.e. analytic for all z
- (CREs are satisfied) $u = e^x \cos y$; $v = e^x \sin y$
  $u_x = e^x \cos y = v_y$
  $u_y = -e_x \sin y = -v_x$
- $(e^z)^/ = e^z$
- $e^{z1+z2} = e^{z1} e^{z2}$
- $|e^{iy}| = |\cos y + i \sin y| = \sqrt{(\cos^2 y + \sin^2 y)} = 1$
- $|e^z| = e^x$
- $\arg e^z = y \pm 2n\pi$ (n = 0,1,2.......)

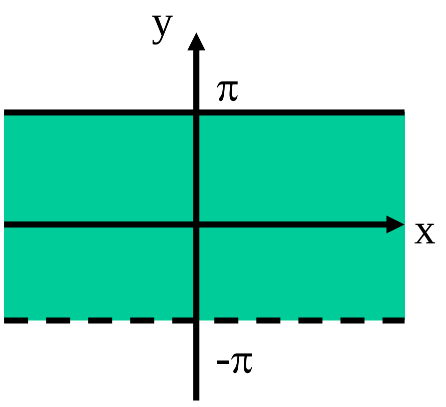
Exponential Function - Properties

- $e^z \neq 0 \ \forall \ z$
- e^z is periodic with period $2\pi \, i$

  $e^{z+2\pi i} = e^z \ \forall \ z$

  Hence all the values that $w = e^z$ can be found in the horizontal strip of width 2π

  $-\pi < y \leq \pi$



This infinite strip is called a ***fundamental*** *region of* e^z

Example

Find all the solutions of $e^z = 3+4i$

Solution

$|e^z| = e^x = 5$

$x = \ln(5) = 1.609$ is the real part

$e^x \cos y = 3; \quad e^x \sin y = 4$

$\Rightarrow \cos y = 0.6; \quad \sin y = 0.8 \quad \Rightarrow \quad y = 0.927$

$\therefore \quad z = 1.609 + 0.927\, i \pm 2n\pi\, i \qquad (n=0,1,2.....)$