ERG 2012B
Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #2: Complex Numbers & Planes

Integer Powers of z

$z = r(\cos\theta + i\sin\theta)$ – Polar Form

$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$z^{-2} = r^{-2}[\cos(-2\theta) + i\sin(-2\theta)]$

more generally

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

Formula of De Moivre:

for $|z| = r = 1$

$$z^n = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Roots of z

If $z = w^n$, then "w" is called the n^{th} root of z written as

$$w = \sqrt[n]{z}$$

and there are "n" distinct values of "w"

- the symbol "$\sqrt[n]{z}$" is ***multi valued***

Let $\quad w = R(\cos\phi + i\sin\phi), \qquad z = r(\cos\theta + i\sin\theta)$

then $\quad w^n = R^n[\cos(n\phi) + i\sin(n\phi)] = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad r = R^n, \qquad \text{or} \qquad R = \sqrt[n]{r}$

and $\quad n\phi = \theta + 2k\pi \quad \text{or} \qquad \phi = \theta/n + 2k\pi/n$ (k is an integer)

- for k = 0,1,...., n-1, we get "n" distinct roots of "w"

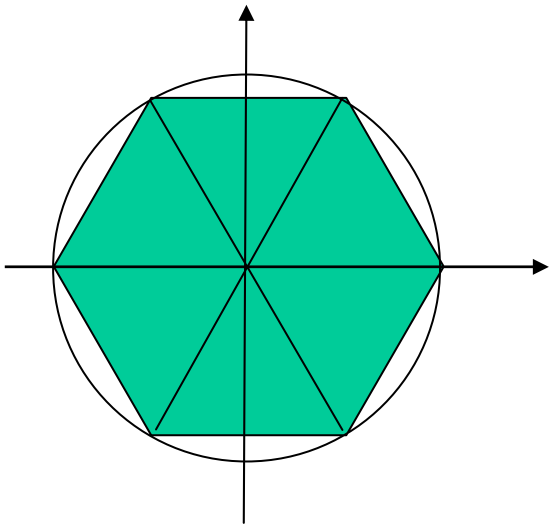
Roots of z

Hence, let $z = r(\cos\theta + i \sin\theta)$, then

$$\sqrt[n]{z} = \sqrt[n]{r}\ \{\cos[(\theta+2k\pi)/n] + i \sin[(\theta+2k\pi)/n]\}$$

where $k = 0,1,\ldots, n-1$ ....(*)

These "n" values lie in a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of "n" sides.



The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg(z) and k=0 in (*) is called the ***principal value*** of $w = \sqrt[n]{z}$

Roots of unity

If r=1 in (*) we get the ***nth* *roots of unity***

$^n\sqrt{1} = \cos[(2k\pi)/n] + i \sin[(2k\pi)/n], \quad k = 0,1,\ldots., n-1$

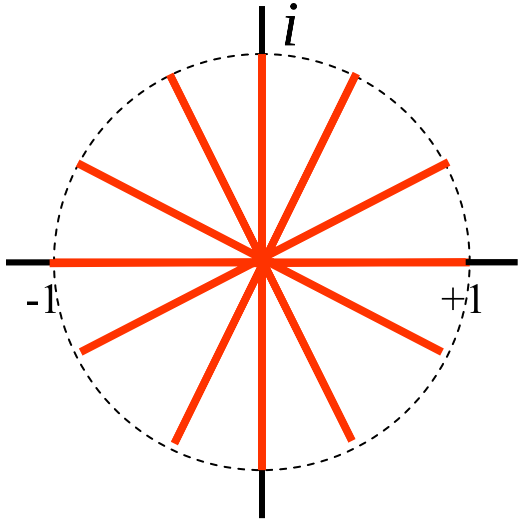
If we denote the values corresponding to k=1 by "w" then the "n" values of $^n\sqrt{1}$ can be written as

$1, w, w^2, w^3, \ldots.., w^{n-1}$

If w_1 is any nth root of z

then the values of $^n\sqrt{z}$ are:

$w_1, w_1 w, w_1 w^2, \ldots\ w_1 w^{n-1}$



n=12

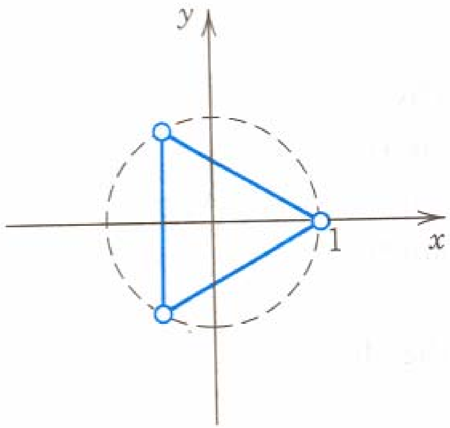


Fig. 297. $\sqrt[3]{1}$

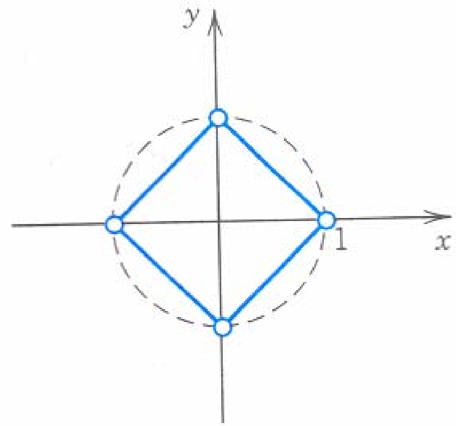


Fig. 298. $\sqrt[4]{1}$

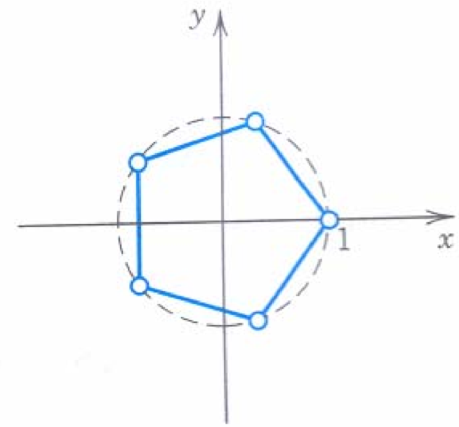


Fig. 299. $\sqrt[5]{1}$

Example – Square Root

If $\quad z = r(\cos\theta + i\sin\theta)$

then $\quad w = \sqrt{z}$ has two values

$\quad w_1 = \sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)]$

and $\quad w_2 = \sqrt{r}\,[\cos(\pi + \theta/2) + i\sin(\pi + \theta/2)]$ $\mathbf{= -w_1}$

Example – Complex Quadratic Equation

Solve

$$z^2 - (5+i)\, z + 8 + i = 0$$

Solution

$$z = [(5+i) \pm \sqrt{(5+i)^2 - 4(8+i)}]/2$$

$$= (5+i)/2 \pm \sqrt{(-2+3i/2)}$$

$$= (5+i)/2 \pm \{\sqrt{(5/2+(-2))/2} + i\sqrt{(5/2-(-2))/2}\}$$

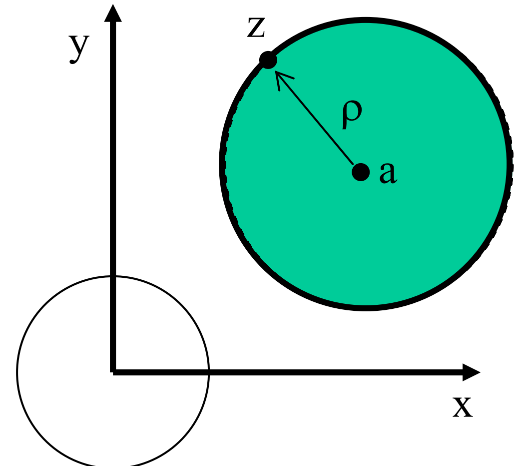
$$= (5+i)/2 \pm [1/2 + 3i/2]$$

$$= \begin{cases} \mathbf{3 + 2}\boldsymbol{i} \\ \mathbf{2 -}\ \boldsymbol{i} \end{cases}$$

Geometry

- The distance between two points "z" and "a" is $|z-a|$
- Hence, a circle "C" of radius "ρ" and center at a is given by:

$$|z-a| = \rho$$



- In particular, the **unit circle** (the circle of radius 1, center at the origin, a=0) is

$$|z| = 1$$

- Points inside the circle "C" are represented by:

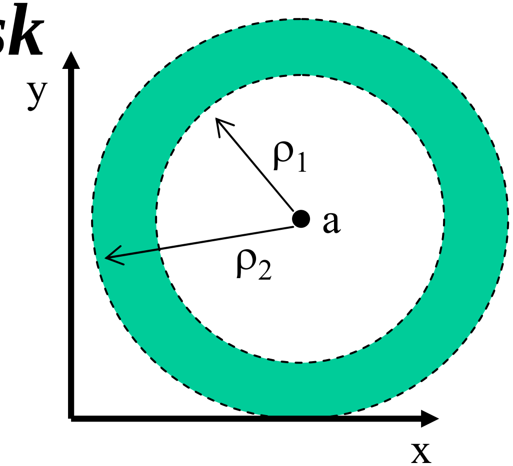
$$|z-a| < \rho$$

- Such a region is called an ***open circular disk*** or a ***neighbourhood*** of the point "a"

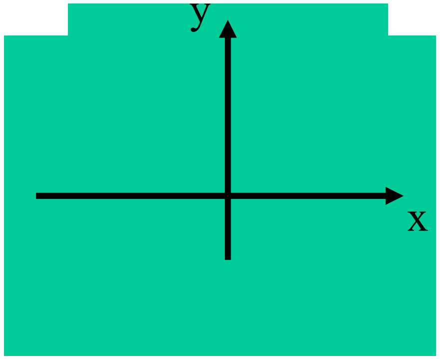
$|z-a| \le \rho$ defines a ***closed circular disk***

- The region between two concentric circles of radii ρ_1 and ρ_2 is given by:

$$\rho 1 < |z-a| < \rho 2$$



Half-planes



(Open) upper half-plane

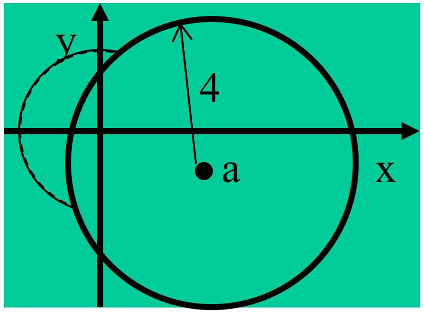
- the set of all points $z = x+iy$ such that $y > 0$
- lower half-plane: $z = x+iy$ such that $y < 0$
- right half-plane: $z = x+iy$ such that $x > 0$
- left half-plane: $z = x+iy$ such that $x < 0$

Example 1

Determine the region in the complex plane given by:

$$|z-3+i| \leq 4$$

Solution



this is a closed circular disk of radius 4 with centre at $3-i$

Example 2

determine the regions: (a) $|z|<1$; (b) $|z| \leq 1$; (c) $|z| > 1$

Solution

(a) Open unit circle; (b) closed unit circle; (c) exterior of unit circle

Sets in the Complex Plane

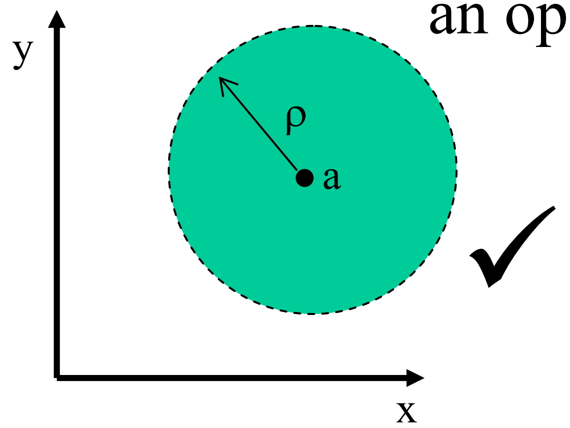
A set of points in the complex planes is ***any*** collection of finite or infinite points. *E.g. solutions of a quadratic equation, points on a line, the points in the interior of a circle, etc*.

A set S is called **open** if every point of S has a neighbourhood consisting entirely of points that belong to S. *e.g. the points in the interior of a circle or a square and the points of the right half plane*.

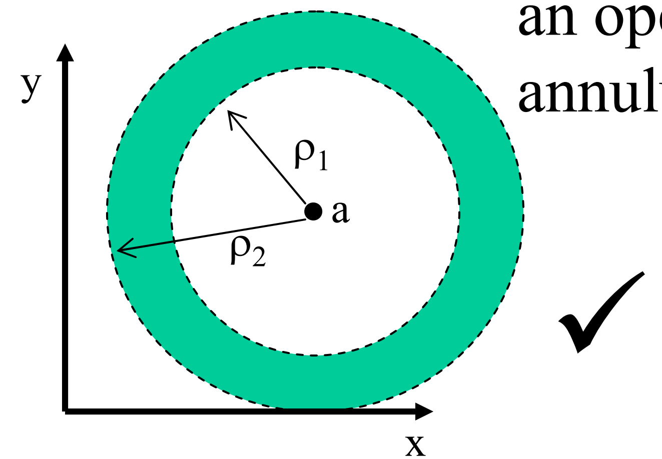
An open set S is called **connected** if any two of its points can be joined by a broken line of finitely many straight line segments all of whose points belong to S. An open connected set is called a **domain**.

Connected Sets

Are the following domains?

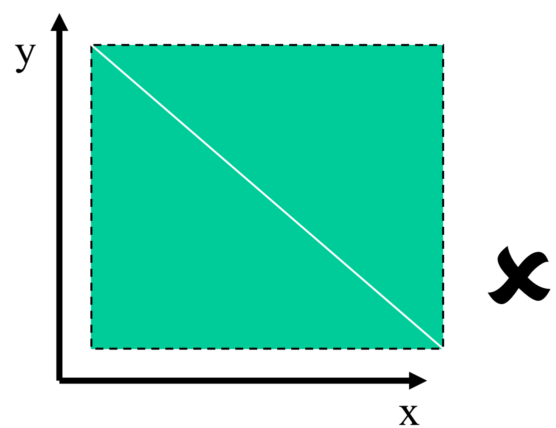


an open disk?



an open annulus?

open square with a diagonal removed?



no path from bottom left to upper right