ERG 2012B

Advanced Engineering Mathematics II

Part III

Introduction to Numerical Methods

Lecture #18

Numerical Method Basics & Interpolation

Secant Method

We obtain the **secant method** from Newton's method if we replace the derivative $f'(x)$ by the difference quotient

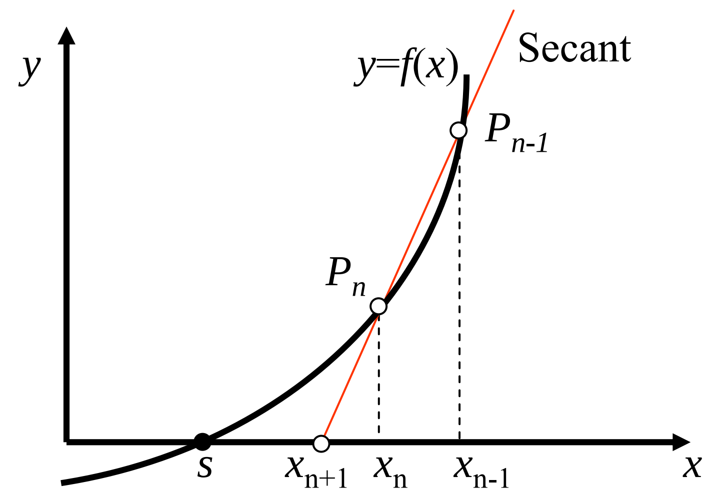
$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Then instead of Newton's method we have:

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

We now need to guess two starting values x_0 and x_1 but avoid the evaluation of derivatives

Geometrically, we intersect the x-axis at x_{n+1} with the secant of $f(x)$ passing through P_{n-1} and P_n



Example 8

Secant method

Find the positive solution of $2 \sin x = x$, starting from $x_0=2$ and $x_1=1.9$

Solution: Secant iteration formula is:

$$x_{n+1} = x_n - \frac{(x_n - 2\sin x_n)(x_n - x_{n-1})}{x_n - x_{n-1} + 2(\sin x_{n-1} - \sin x_n)} = x_n - \frac{N_n}{D_n}$$

Numerical values are:

n	x_n	N_n	D_n	$x_{n+1} - x_n$
0	2.000000			
1	1.900000	-0.000740	-0.174005	-0.004253
2	1.895747	-0.000002	-0.006986	0.000252
3	1.895494	0		0

Bisection Method

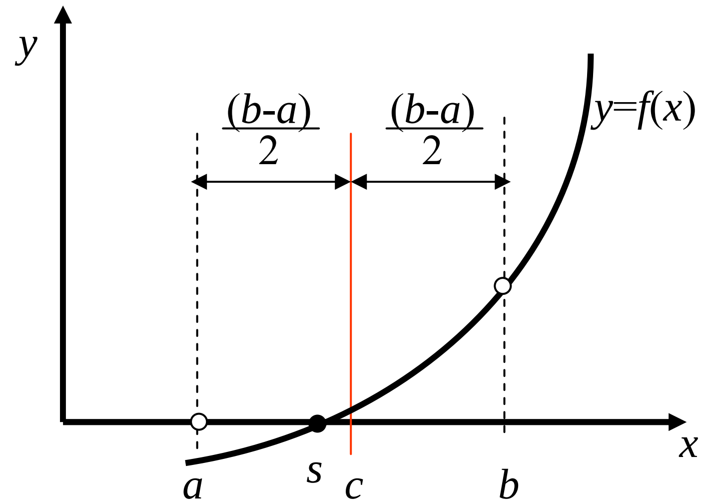
This is a simple but slowly convergent method for finding a solution of $f(x)=0$ with continuous f.

Based on the *intermediate value theorem* – if a continuous function f has opposites signs at $x=a$ and $x=b$ ($>a$) then f must be 0 somewhere between a and b

The solution is found by repeated bisection of the interval into two regions. We then pick the region which still satisfies the sign condition and repeat the exercise.

in example illustration:

if $f(c) < 0$ then
    new region is c,b
elseif $f(c) > 0$ then
    new region is a,c
elseif $f(c) = 0$ then
    solution is c
endif



Method of False Position

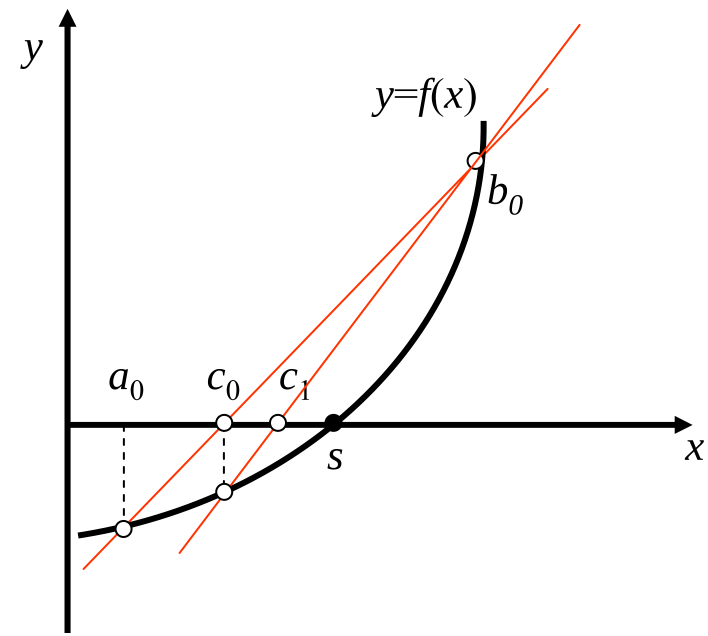
Regula Falsi: The same principle as the bisection method.

http://www.apropos-logic.com/nc/RegulaFalsiAlgorithm.html

We assume that f is continuous.

Compute the x-intercept c_0 of the line through the points

$(a_0, f(a_0))$, $(b_0, f(b_0))$



If $f(c_0) = 0$ then

we are done

If $f(a_0)f(c_0) < 0$ then

set $a_1=a_0$, $b_1=c_0$ and repeat to get c_1 etc..

If $f(a_0)f(c_0) > 0$ (as in example) then

set $a_1=c_0$, $b_1=b_0$ and repeat to get c_1 etc..

Endif

It can be shown that:

$$c_0 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)}$$

Interpolation

Interpolation means to find (approximate) values of a function $f(x)$ for an x **between** *different* x-values, $x_0, x_1, \ldots x_n$ at which the values of $f(x)$ are given.

A standard method is to find a polynomial $p_n(x)$ of degree n (or less) that also has the given values; thus

$$p_n(x_0) = f_0, \;\; p_n(x_1) = f_1, \ldots\ldots, p_n(x_n) = f_n$$

p_n is called an **interpolation polynomial** or **polynomial approximation of *f*** and $x_0, \ldots\ldots, x_n$ the **nodes**

We use p_n to get approximate values of f for x's between x_0 and x_n (**interpolation**) or outside the interval (**extrapolation**)

Existence and Uniqueness: We can always find an n^{th} degree polynomial given n values and that polynomial is unique

Lagrange Interpolation

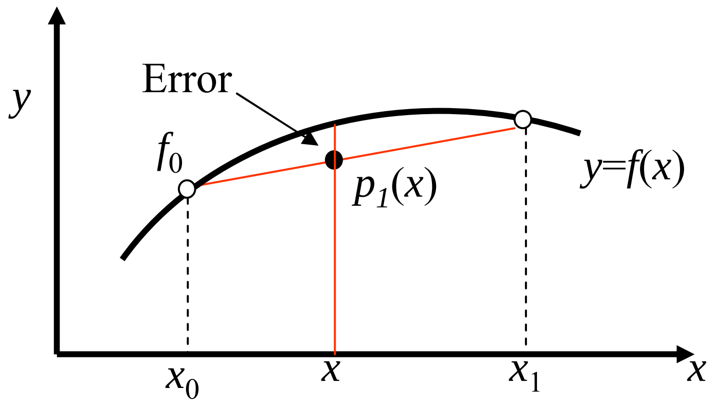
Given (x_0, f_0),......., (x_n, f_n) with arbitrarily spaced x_j, if we multiply each f_j by a polynomial that is 1 at x_j and 0 at the other n nodes and then sum all n+1 polynomials we get a unique interpolation polynomial of degree n or less

Given (x_0, f_0) and (x_1, f_1)

$$\text{Let } L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$\text{then } L_0(x_0) = 1, \; L_0(x_1) = 0, \; L_1(x_0) = 0, \; L_1(x_1) = 1$$

Thus the **linear** Lagrange polynomial is



$$p_1(x) = L_0(x) f_0 + L_1(x) f_1$$

$$= \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} \cdot f_1$$

Quadratic Interpolation

is interpolation of given (x_0, f_0), (x_1, f_1), (x_2, f_2) by a 2nd degree polynomial $p_2(x)$ which by Lagrange's idea is

$$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2$$

with $L_0(x_0) = 1,\ L_1(x_1) = 1,\ L_2(x_2) = 1$ and

$L_0(x_1) = L_0(x_2) = 0$ etc.

Also:

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{l_1(x)}{l_1(x_1)} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{l_2(x)}{l_2(x_2)} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Example 1

Linear Lagrange Interpolation:

Compute ln(9.2) from ln(9.0)=2.1972 and ln(9.5)=2.2513 and determine the error from ln(9.2) = 2.2192 (4D)

Solution: x_0=9.0, x_1=9.5, f_0=ln(9.0), f_1=ln(9.5)

so that:

$$L_0(9.2) = \frac{9.2 - 9.5}{9.0 - 9.5} = 0.6, \quad L_1(9.2) = \frac{9.2 - 9.0}{9.5 - 9.0} = 0.4$$

and we get the answer:

$$\ln(9.2) = p_1(9.2) = L_0(9.2)f_0 + L_1(9.2)f_1$$

$$= 0.6 \text{ x } 2.1972 + 0.4 \text{ x } 2.2513 = 2.2188$$

and the error ε = a – ã = 2.2192 – 2.2188 = 0.0004.

Hence linear interpolation is not sufficient to to get 4D accuracy

Example 2

Quadratic Lagrange Interpolation:

Compute ln(9.2) from ln(9.0)=2.1972, ln(9.5)=2.2513 and ln(11.0) = 2.3979

Solution:

$$L_0(x) = \frac{(x-9.5)(x-11.0)}{(9.0-9.5)(9.0-11.0)} = x^2 - 20.5x + 104.5, \ L_0(9.2) = 0.5400$$

$$L_1(x) = \frac{(x-9.0)(x-11.0)}{(9.5-9.0)(9.5-11.0)} = -\tfrac{1}{0.75}(x^2 - 20x + 99), \ L_1(9.2) = 0.4800$$

$$L_2(x) = \frac{(x-9.0)(x-9.5)}{(11.0-9.0)(11.0-9.5)} = \tfrac{1}{3}(x^2 - 18.5x + 85.5), \ L_2(9.2) = -0.0200$$

and $\ln(9.2) \approx p_2(9.2) = 0.5400\text{x}2.1972+0.4800\text{x}2.2513-0.0200\text{x}2.3979$
$= 2.2192$

Which is exact to 4D

General Lagrange Interpolation

For general n we obtain:

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} L_k(x) f_k = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$$

where:

$$l_0(x) = (x - x_1)(x - x_2)\cdots(x - x_n),$$

$$l_k(x) = (x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n), \qquad 0 < k < n$$

$$l_n(x) = (x - x_0)(x - x_1)\cdots(x - x_{n-1}),$$

Error estimate: the (n+1)th derivative ($f^{(n+1)}$) gives a measure of the error $\varepsilon_n(x) = f(x) - p_n(x)$. It can be shown that this is true if $f^{(n+1)}$ exists and is continuous and that with a suitable t between x_0 and x_n

$$\varepsilon_n(x) = f(x) - p_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(t)}{(n+1)!}$$

Notice: $\varepsilon_n(x)=0$ at the nodes

Example 3

Error estimate of linear interpolation:

Estimate the error in Example 1 by:

$$\varepsilon_1(x) = f(x) - p_1(x) = (x - x_0)(x - x_1)\frac{f''(t)}{2!}$$

Solution: n=1, $f(t)$=ln(t), $f^{/}(t)$ = 1/t, $f^{//}(t)$ = -1/t^2 so that

$$\varepsilon_1(x) = (x - 9.0)(x - 9.5)\frac{-1}{2t^2}$$

$$\varepsilon_1(9.2) = \frac{0.03}{t^2} \qquad \text{where} \quad 9.0 \le t \le 9.5$$

so that the maximum is $0.03/9^2 = 0.00037$ and the minimum is $0.03/9.5^2 = 0.00033$ so that

$0.00033 \le \varepsilon \le 0.0003$**8** (as $0.3/81 = 0.0003703 > 0.00037$)

But error *calculated* in example 1 was $0.0004 > 0.00038$.

If example 1 repeated with 5D we get $\varepsilon = 0.00035$

Newton's Divided Difference Interpolation

Let $p_{n-1}(x)$ be the $(n\text{-}1)^{\text{th}}$ *Newton polynomial* (we will determine the form later) so that: $p_{n-1}(x_0) = f_0, \ldots, p_{n-1}(x_{n-1}) = f_{n-1}$.

And we will write the n^{th} Newton polynomial as:

$$p_n(x) = p_{n-1}(x) + g_n(x)$$

with

$$g_n(x) = p_n(x) - p_{n-1}(x)$$

so that $p_n(x_0) = f_0, \ldots, p_n(x_n) = f_n$

Since p_n and p_{n-1} agree at $x_0, \ldots\ldots x_{n-1}$ we see that g_n is zero there. Also g_n will generally be a polynomial of n^{th} degree as p_n is and p_{n-1} can be of degree n-1 at most. Hence g_n must be of the form:

$$g_n(x) = a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

We can determine the constant a_n as follows:

Newton's Divided Difference Interpolation

We set $x = x_n$ and solve $g_n(x) = a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$

and substitute $g_n(x_n) = p_n(x_n) - p_{n-1}(x_n)$ and $p_n(x_n) = f_n$

gives $$a_n = \frac{f_n - p_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}$$

Thus a_k equals the **kth divided difference**, recursively denoted and defined as

$$a_1 = f[x_0, x_1] = \frac{f_1 - f_0}{(x_1 - x_0)}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

and in general:

$$a_k = f[x_0, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{(x_k - x_0)}$$

Newton's Divided Difference Interpolation

So that the k^{th} Newton polynomial becomes:

$$p_k(x_n) = p_{k-1}(x_n) + f[x_0, \cdots, x_k](x - x_0)(x - x_1)\cdots(x - x_{k-1})$$

with $p_0(x) = f_0$. Then by repeated application with $k = 1,\ldots n$ this finally gives **Newton's divided difference interpolation formula:**

$$f(x) \approx f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$
$$+ \cdots + (x - x_0)\cdots(x - x_{n-1}) f[x_0, \cdots, x_n]$$

Which looks more complicated than it really is. It is quite easy to write as a computer program - see text book.

Example 4

Compute $f(9.2)$ from the given values

	Difference Table			
x_j	$f_j = f(x_j)$	$f[x_j, x_{j+1}]$	$f[x_j, x_{j+1}, x_{j+2}]$	$f[x_j, \ldots, x_{j+3}]$
8.0	2.079442			
		0.117783		
9.0	2.197225		-0.006433	
		0.108134		0.000411
9.5	2.251292		-0.005199	
		0.097735		
11.0	2.397895			

given values

we use the shaded numbers in the polynomial so that:

$$f(x) \approx p_3(x) = 2.079422 + 0.117783(x-8.0) - 0.006433(x-8.0)(x-9.0)$$
$$+ 0.000411(x-8.0)(x-9.0)(x-9.5)$$

At x=9.2

$$f(9.2) \approx 2.079422 + 0.141340 - 0.001544 - 0.000030 = 2.219208$$