ERG 2012B
Advanced Engineering Mathematics II

Part II: Linear Algebra

Lecture #16

Eigen-pairs and Special Matrices

Unitary Matrices

We can also extend the properties of orthogonal matrices with real parameters to Unitary matrices with complex parameters.

Instead of the real vector space R^n of all real vectors with n components and real numbers as scalars, we use the **complex vector space** $\boldsymbol{C^n}$ of all complex vectors with n complex numbers as components and complex numbers as scalars.

Now the **inner product** is defined by:

$$\mathbf{a} \bullet \mathbf{b} = \overline{\mathbf{a}}^{\mathrm{T}} \mathbf{b}$$

and the **length** or **norm** of a vector in C^n given by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \bullet \mathbf{a}} = \sqrt{\overline{\mathbf{a}}^{\mathrm{T}} \mathbf{a}} = \sqrt{\overline{a}_1 a_1 + \ldots + \overline{a}_n a_n} = \sqrt{|a_1|^2 + \ldots + |a_n|^2}$$

If the vectors are real this relaxes to the inner product as defined with orthogonal matrices.

Inner Product

Theorem 2: A **Unitary transformation**, that is $\mathbf{y} = \mathbf{A}\mathbf{x}$ with a unitary matrix $\mathbf{A}$ preserves the value of the inner product, hence also the norm.

Proof: is the same as previously with orthogonal matrices, except we now use complex conjugates:

$$\mathbf{u} \bullet \mathbf{v} = \overline{\mathbf{u}}^{T}\mathbf{v} = (\overline{\mathbf{A}}\overline{\mathbf{a}})^{T}\mathbf{A}\mathbf{b} = \overline{\mathbf{a}}^{T}\overline{\mathbf{A}}^{T}\mathbf{A}\mathbf{b} = \overline{\mathbf{a}}^{T}\mathbf{I}\mathbf{b} = \overline{\mathbf{a}}^{T}\mathbf{b} = \mathbf{a} \bullet \mathbf{b}$$

The complex analog of an *orthonormal system* of real vectors is a **unitary system**, defined by:

$$\mathbf{a}_j \bullet \mathbf{a}_k = \mathbf{a}_j^{T}\mathbf{a}_k = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases}$$

Theorem 3: A square matrix is unitary iff its column vectors (and row vectors) form a unitary system

Proof: Same as for orthonormal system previously

Determinant of Unitary Matrix

Theorem 4: The determinant of an unitary matrix has absolute value 1

Proof: similar to previous case for orthogonal matrix

$$\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B} \quad \text{and}$$

$$\det \mathbf{A}^{\mathbf{T}} = \det \mathbf{A}$$

$$1 = \det \mathbf{I} = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A}\bar{\mathbf{A}}^{\mathbf{T}}) = \det \mathbf{A} \det \bar{\mathbf{A}}^{\mathbf{T}} = \det \mathbf{A} \det \bar{\mathbf{A}}$$

$$= \det \mathbf{A}\, \overline{\det \mathbf{A}} = |\det \mathbf{A}|^2$$

Example 5: For the vectors $\mathbf{a}^{\mathbf{T}}=[1 \quad i]$ and $\mathbf{b}^{\mathbf{T}} = [3i \quad 2+i]$ we get

$$\bar{\mathbf{a}}^{\mathbf{T}}\mathbf{b} = 3i - i(2+i) = 1+i$$

then with

$$\mathbf{A} = \begin{bmatrix} 0.6i & 0.8 \\ 0.8 & 0.6i \end{bmatrix} \quad \text{then} \quad \mathbf{Aa} = \begin{bmatrix} 1.4i \\ 0.2 \end{bmatrix} \quad \text{and} \quad \mathbf{Ab} = \begin{bmatrix} -0.2+0.8i \\ -0.6+3.6i \end{bmatrix}$$

This gives $(\bar{\mathbf{A}}\bar{\mathbf{a}})^{\mathbf{T}}\mathbf{Ab}=1+i$ illustrating Theorem 2. The matrix is unitary. Its columns form a unitary system and $\det \mathbf{A} = -1$.

Similarity of Matrices

An n x n matrix $\hat{\mathbf{A}}$ is called **similar** to an n x n matrix $\mathbf{A}$ if

$$\hat{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

for some (nonsingular) n x n matrix $\mathbf{T}$. This transformation, which gives $\hat{\mathbf{A}}$ from $\mathbf{A}$, is called a **similarity transformation**

Theorem 1: Eigenvalues and eigenvectors of similar matrices

- If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$.
- If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{T}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ for the same eigenvalue

Proof: From $\mathbf{Ax} = \lambda\mathbf{x}$ we get $\mathbf{T}^{-1}\mathbf{Ax} = \lambda\mathbf{T}^{-1}\mathbf{x}$. Also $\mathbf{I} = \mathbf{T}\mathbf{T}^{-1}$, so:

$$\mathbf{T}^{-1}\mathbf{Ax} = \mathbf{T}^{-1}\mathbf{AIx} = \mathbf{T}^{-1}\mathbf{ATT}^{-1}\mathbf{x} = \hat{\mathbf{A}}(\mathbf{T}^{-1}\mathbf{x}) = \lambda\mathbf{T}^{-1}\mathbf{x}$$

Hence λ is also an eigenvalue of $\hat{\mathbf{A}}$ and $\mathbf{T}^{-1}\mathbf{x}$ a corresponding eigenvector

Linear Independence

Theorem 2: Linear independence of eigenvectors

Let $\lambda_1, \lambda_2,\ldots,\lambda_k$ be eigenvalues of an n x n matrix. Then the eigenvectors $\mathbf{x_1}, \mathbf{x_2},\ldots,\mathbf{x}_k$ form a linearly independent set.

Proof: Let the conclusion be false. Let r be the largest integer such that $\{\mathbf{x_1},\ldots,\mathbf{x}_r\}$ is a linearly independent set.

Thus there are scalars $c_1,\ldots,c_{r+1}$, not all zero, such that:

$$c_1\mathbf{x_1} + \ldots + c_{r+1}\mathbf{x}_{r+1} = 0$$

Multiply both sides by $\mathbf{A}$ and use $\mathbf{Ax}_j = \lambda_j\mathbf{x}_j$ we obtain:

$$c_1\lambda_1\mathbf{x_1} + \ldots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0$$

remove last term by subtracting λ_{r+1} times original from this:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x_1} + \ldots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

Then $c_1(\lambda_1 - \lambda_{r+1})=0,\ldots\ c_r(\lambda_r - \lambda_{r+1})=0$ since linearly independent. Hence $c_1=\ldots=c_r=0$ since eigenvectors are distinct. Thus $c_{r+1}\mathbf{x}_{r+1}=0$ and thus $c_{r+1}=0$ ($\mathbf{x}_{r+1}\neq 0$). But original assumption was that not all scalars are zero. So assumption is wrong.

Basis of Eigenvectors

Theorem 3: Basis of eigenvectors

If an n x n matrix **A** has n distinct eigenvalues, then **A** has a basis of eigenvectors for C^n (or R^n)

Example 1: The matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \quad \text{has a basis of eigenvectors} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

corresponding to the eigenvalues $\lambda_1 = 8$, $\lambda_2 = 2$ (from previous example)

Example 2

Example 2: Basis when not all eigenvalues are distinct

Even if not all n eigenvalues are different, a matrix **A** may still provide a basis of eigenvectors for C^n.

e.g.

Example 2

Multiple Eigenvalues

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution The characteristic determinant gives the characteristic equation: $-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

The roots (eigenvalues) are $\lambda_1=5$, $\lambda_2=\lambda_3=-3$

To find the eigenvectors use Gauss elimination on $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ letting $\lambda=\mathbf{5}$ and then $\lambda=-3$:

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \ \& \ \mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

However, **A** may not have enough linearly independent eigenvectors to make up a basis:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ has only one eigenvector $$\begin{bmatrix} k \\ 1 \end{bmatrix}$$

So **A** does not provide a basis of eigenvectors for R^2

Diagonalization of a Matrix

Theorem 4: Basis of eigenvectors

A Hermitian, skew-Hermitian or unitary matrix has a basis of eigenvectors for C^n that is a unitary system.

Theorem 5: Diagonalization of a matrix

If an n x n matrix $\mathbf{A}$ has a basis of eigenvectors then

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is diagonal, with the eigenvalues of $\mathbf{A}$ as the main entries on the main diagonal. $\mathbf{X}$ is the matrix with these eigenvectors as column vectors. Also

$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$$

Proof: Let $\mathbf{x}_1..\mathbf{x}_n$ be a basis of $\mathbf{A}$ for C^n with eigenvalues $\lambda_1..\lambda_n$

Then $\mathbf{X}=[\mathbf{x}_1...\mathbf{x}_n]$ has rank n. Hence $\mathbf{X}^{-1}$ exists and as $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$\mathbf{AX} = \mathbf{A}[\mathbf{x}_1 \ldots \mathbf{x}_n] = [\mathbf{A}\mathbf{x}_1 \ldots \mathbf{A}\mathbf{x}_n] = [\lambda_1\mathbf{x}_1 \ldots \lambda_n\mathbf{x}_n]$$

so that $\mathbf{AX}=\mathbf{XD}$. Hence $\mathbf{X}^{-1}\mathbf{AX} = \mathbf{D}$

Note: $\mathbf{D}^2=\mathbf{DD}=\mathbf{X}^{-1}\mathbf{AXX}^{-1}\mathbf{AX}=\mathbf{X}^{-1}\mathbf{AAX}=\mathbf{X}^{-1}\mathbf{A}^2\mathbf{X}$

Diagonalization

Example 4: We can show that

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \text{ has eigenvectors } \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \text{ Hence } \mathbf{X} = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$$

so that:

$$\mathbf{X^{-1}AX} = \frac{1}{-5}\begin{bmatrix} -1 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & -0.8 \end{bmatrix}\begin{bmatrix} 24 & 1 \\ 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

where the eigenvalues of **A** can be found to be 6 and 1

Diagonalization

Example 5: Diagonalize

$$\mathbf{A}=\begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

Solution: the characteristic determinant gives the characteristic equation $-\lambda^3-\lambda^2+12\lambda=0$. The roots (eigenvalues) are $\lambda_1=3$, $\lambda_2=-4$, $\lambda_3=0$. Apply Gauss elimination to $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ with $\lambda=\lambda_1, \lambda_2, \lambda_3$ we find the eigenvectors and then $\mathbf{X}^{-1}$ by Gauss-Jordan method:

$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{X}=\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \ \mathbf{X}^{-1}=\begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Calculating **AX** and multiplying by $\mathbf{X}^{-1}$ from the left we obtain:

$$\mathbf{D}=\mathbf{X}^{-1}\mathbf{A}\mathbf{X}=\begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}\begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix}=\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Transformation of Forms

If we have a quadratic form Q then $Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a real symmetric matrix and $\mathbf{x}$ are eigenvectors of $\mathbf{A}$.

$\mathbf{A}$ has a basis of n eigenvectors and the matrix $\mathbf{X}$ of these vectors as column vectors is orthogonal so that $\mathbf{X}^{-1}=\mathbf{X}^T$.

Thus $\mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{X}\mathbf{D}\mathbf{X}^T$

and $Q = \mathbf{x}^T\mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x}$

If we let $\mathbf{X}^T\mathbf{x} = \mathbf{y}$, then since $\mathbf{X}^T = \mathbf{X}^{-1}$ we have $\mathbf{x} = \mathbf{X}\mathbf{y}$

and Q becomes $Q = \mathbf{y}^T\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots\ldots + \lambda_n y_n^2$ (*)

This proves the following Theorem:

Theorem 6 (Principal Axes Theorem)

The substitution $\mathbf{x} = \mathbf{X}\mathbf{y}$ transforms a quadratic form

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$$

symmetric

to the principal axes form (*), where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of the matrix $\mathbf{A}$, and $\mathbf{X}$ is the orthogonal matrix of eigenvectors.

Example Conic Sections

Example 6: Find out what type of conic section the following quadratic form represents and transform it to principal axes:

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Solution: We have $Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$, where

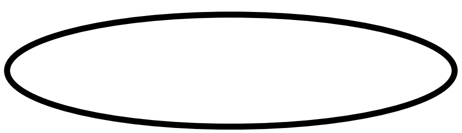
$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This gives the characteristic equation $(17-\lambda)^2-15^2=0$, which has roots $\lambda_1=2$, $\lambda_2=32$. So that (*) becomes

$$Q = 2y_1^2+32y_2^2$$

We can see from this that Q=128 represents the ellipse

$2y_1^2+32y_2^2=128$ or $y_1^2/8^2 + y_2^2/2^2 = 1$



Example Conic Sections

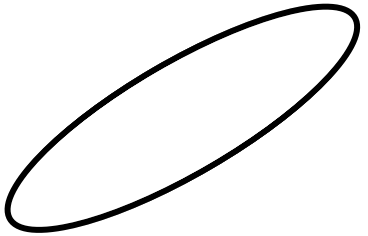
Example 6: Find out what type of conic section the following quadratic form represents and transform it to principal axes:

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Solution: (Continued)

To determine the direction of the principal axes in x_1x_2 coords we determine normalized eigenvectors from $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with λ=2 and 32 and use $\mathbf{x} = \mathbf{X}\mathbf{y}$. Then

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$



hence:

$$\mathbf{x} = \mathbf{X}\mathbf{y} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \begin{aligned} x_1 &= y_1/\sqrt{2} - y_2/\sqrt{2} \\ x_2 &= y_1/\sqrt{2} + y_2/\sqrt{2} \end{aligned}$$

This is a 45^{o} rotation