ERG 2012B

Advanced Engineering Mathematics II

Part II: Linear Algebra

Lecture #14

Inverse, Determinant and Eigen-pairs

Some Useful Formulas for Inverses

For a nonsingular 2x2 matrix we obtain

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where det $\mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$ and will be discussed later - it is simple to observe that the formula holds

For a nonsingular diagonal matrix the entries of $\mathbf{A}^{-1}$ on the main diagonal are reciprocals of those of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & \cdots & 0 \\ & \cdots & \\ 0 & & \frac{1}{a_{nn}} \end{bmatrix}$$

The inverse of the inverse is the given matrix $\mathbf{A}$

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

Examples

Example: Inverse of a 2x2 matrix: [det $\mathbf{A}$ = $a_{11}a_{22}$-$a_{12}a_{21}$]

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

Example: Inverse of a diagonal matrix:

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of a Product

The inverse of a product **AC** can be calculated by inverting each factor separately and multiplying the results in *reverse order:*

$$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$$

Proof: $\mathbf{AC}(\mathbf{AC})^{-1} = \mathbf{I}$

$\mathbf{C}(\mathbf{AC})^{-1} = \mathbf{A}^{-1}$

$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$ (remember order is important)

In general with more than two matrices:

$$(\mathbf{AC}\ldots\ldots\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\ldots\mathbf{C}^{-1}\mathbf{A}^{-1}$$

Cancellation Law

Theorem 2: Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be n x n matrices. Then:

(a) If rank $\mathbf{A} = n$ and $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$

(b) If rank $\mathbf{A} = n$ then $\mathbf{AB} = \mathbf{0}$ implies $\mathbf{B} = \mathbf{0}$. Hence if $\mathbf{AB} = \mathbf{0}$, but $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$ then rank $\mathbf{A} < n$ and rank $\mathbf{B} < n$

(c) If $\mathbf{A}$ is singular, so are $\mathbf{AB}$ and $\mathbf{BA}$

Proof: (a) Premultiply $\mathbf{AB} = \mathbf{AC}$ on both sides by $\mathbf{A}^{-1}$ which exists by previous theorem

(b) Premultiply $\mathbf{AB} = \mathbf{0}$ on both sides by $\mathbf{A}^{-1}$

(c) Rank $\mathbf{A} < n$ if $\mathbf{A}$ is singular. Hence $\mathbf{Ax} = \mathbf{0}$ has nontrivial solutions. Multiplication gives $\mathbf{BAx} = \mathbf{0}$. Hence same solutions satisfy $\mathbf{BAx} = \mathbf{0}$. So rank $\mathbf{BA} < n$ and $\mathbf{BA}$ is singular

Also $\mathbf{A}^T$ is singular, hence $\mathbf{B}^T\mathbf{A}^T$ is singular (above). But $\mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T$, hence $\mathbf{AB}$ is singular as well

Determinants

Second-Order Determinants

A determinant of second order is denoted and defined by

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The definition is suggested by systems like:

(a) $a_{11}x_1 + a_{12}x_2 = b_1$

(b) $a_{21}x_1 + a_{22}x_2 = b_2$

The solution of which can be written $x_1 = D_1/D$, $x_2 = D_2/D$ with D as above and

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1a_{22} - a_{12}b_2 \qquad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = a_{11}b_2 - b_1a_{21}$$

Provided $D \neq 0$; This is called **Cramer's rule**

Example

Use of Second-Order Determinants

If

$$4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

then

$$D = \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = 14 \qquad D_1 = \begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix} = 84 \qquad D_2 = \begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix} = -56$$

so that $x_1 = 84/14 = 6$ and $x_2 = -56/14 = -4$

Determinants

Third-Order Determinants

A determinant of third order can be defined by

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Note: The signs on the right are + - +. Each of these 3 terms is an entry in the first column of D times its **minor** ie the 2nd order determinant obtained by deleting the row and column of that entry from D

Expanding out the minors we get:

$$D = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}$$

Determinants

Third-Order Determinants

For linear systems of three equations in three unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Cramer's rule is

$$x_1 = D_1/D, \quad x_2 = D_2/D, \quad x_3 = D_3/D$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Determinants of Any Order n

A determinant of order n is a scalar associated with an n x n matrix and is written:

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Cofactor of a_{jk} in *D*

Minor of a_{jk} in *D*

and is defined for $n = 1$ by $D = a_{11}$ and for $n \geq 2$ by

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \ldots + a_{jn}C_{jn} \qquad (j = 1,2,\ldots, \text{ or } n)$$

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \ldots + a_{nk}C_{nk} \qquad (k = 1,2,\ldots, \text{ or } n)$$

where

$$C_{jk} = (-1)^{j+k} M_{jk}$$

***D* determined by any row or column**

and M_{jk} is a determinant of order n-1 – the determinant of the submatrix obtained from **A** by deleting the row and column of the entry a_{jk} – as before

Example 2

Determinant of second order

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

formula for D gives four possible expansions:

by first row: $D = a_{11}a_{22} + a_{12}(-a_{21})$

by second row: $D = a_{21}(-a_{12}) + a_{22}a_{11}$

by first column: $D = a_{11}a_{22} + a_{21}(-a_{12})$

by second column: $D = a_{12}(-a_{21}) + a_{22}a_{11}$

All of which are the same value $D = a_{11}a_{22} - a_{12}a_{21}$ stated earlier

Example 3

Minors and cofactors of a third-order determinant

The 3rd order determinant:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

has minors:

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \quad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \quad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \quad M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 3

Minors and cofactors of a third-order determinant

The 3rd order determinant:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

has cofactors:

$$\begin{array}{lll} C_{11} = +M_{11}, & C_{12} = -M_{12}, & C_{13} = +M_{13} \\ C_{21} = -M_{21}, & C_{22} = +M_{22}, & C_{23} = -M_{23} \\ C_{31} = +M_{31}, & C_{32} = -M_{32}, & C_{33} = +M_{33} \end{array}$$

Notice the signs form a checkerboard pattern:

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

Example 4

A third-order determinant

Let

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

The expansion by the first row is:

$$D = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 1(12-0) - 3(4+4) = -12$$

And by the third column is:

$$D = -4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = -4(0+3) + 2(6-6) = -12$$

Example 5

Determinant of a Triangular Matrix

The determinant of any triangular matrix equals the product of all the entries on the main diagonal. To see this expand by rows if the matrix is lower triangular and by columns if it is upper triangular.

E.g.

$$\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3\begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60$$

General Properties of Determinants

Theorem 1 (Transposition)

The value of a determinant is not altered if its rows are written as columns in the same order

Example 6 Transposition

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 0 \\ 0 & 4 & 2 \end{vmatrix} = -12$$

General Properties of Determinants

Theorem 2 (Multiplication by a constant)

If all the entries in one row (or one column) of a determinant are multiplied by the same factor k, the value of the new determinant is k times the value of the given determinant

Proof: Expand the determinant by that row (or column) whose entries are multiplied by k

Caution: $\det k\mathbf{A} = k^n \det \mathbf{A}$ (**not** $k \det \mathbf{A}$)

make sure you understand why.....

Example 7

Application of Theorem 2

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & 0 \\ 1 & 3 & 2 \\ -1 & 0 & 2 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{vmatrix} = 12\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -12$$

From this taking k=0 or directly expanding:

Theorem 3: If all the entries in a row (or a column) of a determinant are zero, the value of the determinant is zero

Example 8

Theorem 4: If each entry in a row (or column) of a determinant is expressed as a binomial, the determinant can be written as the sum of two determinants

Proof: Expand the determinant by the row (or column) whose entries are binomials......

Example 8

$$\begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Interchange of Rows or Columns

Theorem 5: If any two rows (or two columns) of a determinant D are interchanged, the value of D is multiplied by -1

Proof: The proof is by induction. We see that the theorem holds for determinants of order 2 and we can show that it holds for determinants of order n provided it holds for order n-*1*

Let D be of order n and E be obtained from D by interchanging two rows. Expand D and E by a row that is not one of those interchanged – the jth row. Then:

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}, \quad E = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} N_{jk}$$

where N_{jk} is obtained from the minor M_{jk} of a_{jk} in D by interchanging two rows. Since these minors are of order n-1, the induction hypothesis applies and so N_{jk}=-M_{jk} and E=-D

Example 9

Interchange of two rows

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 12$$

Proportional Rows or Columns

Theorem 6: If corresponding entries in two rows (or 2 columns) of a determinant D are proportional, the value of D is zero

Proof: Let the entries in the ith and jth rows of D be proportional, i.e. $a_{jk} = ca_{jk}$, $k = 1,\ldots, n$.

If $c = 0$ then $D = 0$.

If $c \neq 0$ then $D = cB$, where the ith and jth rows of B are identical. Interchange these rows. Then B becomes $-B$. But if the rows are identical the new determinant must still be B as well.
Thus $B = -B$, $B = 0$ and $D = 0$.

Example 10:

$$\begin{vmatrix} 3 & 6 & -4 \\ 1 & -1 & 3 \\ -6 & -12 & 8 \end{vmatrix} = 0$$

More Properties

Theorem 7: Addition of a row or column. The value of a determinant is left unchanged if the entries in a row (or column) are altered by adding to them any constant multiple of the corresponding entries in any other row (or column)

Proof: Apply Theorem 4 (binomial sequence) to the determinant that results from the given addition. This yields a sum of two determinants; one is the original determinant and the other contains two proportional rows – by theorem 6 the second one is zero.

Theorem 8: Determinant of a product of matrices

For any n x n matrices **A** and **B**

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}$$

Derivative

Theorem 9: Derivative of a determinant

The derivative D' of a determinant D of order n whose entries are differentiable functions can be written:

$$D' = D_{(1)} + D_{(2)} + \ldots\ldots + D_{(n)}$$

where $D_{(j)}$ is obtained from D by differentiating the entries in the jth row.

Example

$$\frac{d}{dx}\begin{vmatrix} f & g & h \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} f' & g' & h' \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ p' & q' & r' \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ p & q & r \\ u' & v' & w' \end{vmatrix}$$

Evaluation of Determinant

Example 11: Evaluation of a determinant by reduction to triangular form:

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

Evaluation of Determinant

Example 11: Evaluation of a determinant by reduction to triangular form:

$$D = \begin{vmatrix} \boxed{2} & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \begin{matrix} \\ \text{row2 - 2 row1} \\ \\ \text{row 4 + 1.5 row 1} \end{matrix}$$

$$= \boxed{2} \text{x} \begin{vmatrix} 5 & 9 & -12 \\ 2 & 6 & -1 \\ 8 & 3 & 10 \end{vmatrix}$$

Evaluation of Determinant

Example 11: Evaluation of a determinant by reduction to triangular form:

$$D = \begin{vmatrix} \boxed{2} & 0 & -4 & 6 \\ 0 & \boxed{5} & 9 & -12 \\ 0 & 0 & \boxed{2.4} & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \begin{matrix} \\ \\ \text{row 3 - 0.4 row 2} \\ \text{row 4 – 1.6 row 2} \end{matrix}$$

$$= \boxed{2} \text{x} \boxed{5} \text{x} \begin{vmatrix} 2.4 & 3.8 \\ -11.4 & 29.2 \end{vmatrix}$$

Evaluation of Determinant

Example 11: Evaluation of a determinant by reduction to triangular form:

$$D = \begin{vmatrix} \boxed{2} & 0 & -4 & 6 \\ 0 & \boxed{5} & 9 & -12 \\ 0 & 0 & \boxed{2.4} & 3.8 \\ 0 & 0 & 0 & \boxed{47.25} \end{vmatrix} \text{row } 4 + 4.75 \text{ row } 3$$

$$= \boxed{2 \text{ x } 5 \text{ x } 2.4 \text{ x } 47.25} = 1134$$

Rank in Terms of Determinants

Theorem 1: An m x n matrix $\mathbf{A} = [a_{jk}]$ has rank $r \geq 1$ iff **A** has an r x r submatrix with nonzero determinant, whereas the determinant of every square submatrix with $r+1$or more rows is zero.

In particular, if **A** is a square matrix, **A** is nonsingular, so that the inverse $\mathbf{A}^{-1}$ exists iff det $\mathbf{A} \neq 0$

Proof: Key lies in the fact that elementary row operations do not alter the rank or the property of a determinant being zero or not zero.

Also remember that elementary row operations enable manipulation of matrix to make it easier to see the rank by reducing the matrix to its echelon form – the “shape” of which is described in the theorem.

Cramer's Theorem

Theorem 2: (a) If the determinant D = det **A** of a linear system of n equations

$$
\begin{aligned}
a_{11}.x_1 + \cdots + a_{1n}.x_n &= b_1 \\
a_{21}.x_1 + \cdots + a_{2n}.x_n &= b_2 \\
\cdots\cdot \\
a_{n1}.x_1 + \cdots + a_{nn}.x_n &= b_n
\end{aligned}
\qquad (1)
$$

in the same number of unknowns is not zero then the system has precisely one solution. Which is given by the formulas:

$$x_1 = D_1/D,\ x_2 = D_2/D,\ \ldots\ldots\ x_n = D_n/D$$

as before.

(b) Hence if (1) is **homogeneous** and $D\neq 0$, it has only the trivial solution $x_1=0$, $x_2=0$,...$x_n=0$. If $D=0$, the homogeneous system also has nontrivial solutions

Inverse of Matrix

Theorem 3: The inverse of a nonsingular n x n matrix $\mathbf{A} = [a_{jk}]$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\left[C_{jk}\right]^{\mathrm{T}} = \frac{1}{\det \mathbf{A}}\begin{vmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{vmatrix}$$

where C_{jk} is the cofactor of a_{jk} in det $\mathbf{A}$. Note well that in $\mathbf{A}^{-1}$ we use the transpose of the cofactors – i.e. the cofactor C_{jk} occupies the same place as a_{kj} (**not** a_{jk}) does in $\mathbf{A}$.

Example

Find the inverse of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$\mathbf{A}^{-1} = 0.1\begin{bmatrix} \boxed{-7} & 2 & 3 \\ \boxed{-13} & -2 & 7 \\ \boxed{8} & 2 & -2 \end{bmatrix}$$

Solution: The inverse is: 1/(det **A**) times the matrix of cofactors transposed. det **A** = -1(-4-3)-1(12+1)+2(9-1) = 10

Alternatively:

det **A** = (-1)(-1)4+1x1x(-1)+2x3x3-(-1)(-1)2-1(-1)3-1x3x4 = 10

cofactors:

$$\boxed{C_{11}} = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \boxed{-7} \quad \boxed{C_{12}} = -\begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix} = \boxed{-13} \quad \boxed{C_{13}} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \boxed{8}$$

$$C_{21} = -\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 2 \quad C_{22} = \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix} = -2 \quad C_{23} = \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix} = 2$$

$$C_{31} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = 3 \quad C_{32} = -\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = 7 \quad C_{33} = \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} = -2$$

Eigenvalues and Eigenvectors

If $\mathbf{A}=[a_{jk}]$ is a given n x n matrix, consider the vector equation:

$$\mathbf{Ax} = \lambda\mathbf{x}$$

where λ is a number. It is clear that the zero vector $\mathbf{x}=\mathbf{0}$ is a solution to this for any value of λ.

- A value of λ for which there is a solution $\mathbf{x}\neq\mathbf{0}$ is called an **eigenvalue** or **characteristic value** (or *latent root*)
- The corresponding solutions $\mathbf{x}\neq\mathbf{0}$ themselves are called **eigenvectors** or **characteristic vectors** of **A**
- The set of eigenvectors is called the **spectrum of A**.
- The largest of the absolute values of the eigenvalues is called the **spectral radius of A**
- The set of all eigenvectors corresponding to an eigenvalue of **A,** together with **0,** form a vector space - the **eigenspace.**

Example 1

Determination of eigenvalues and eigenvectors

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solution: eigenvalues. These **must** be determined **first**.

$$\mathbf{Ax} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = \lambda \begin{bmatrix} x_a \\ x_b \end{bmatrix}$$

or

$$\begin{aligned} -5x_a + 2x_b &= \lambda x_a \\ 2x_a - 2x_b &= \lambda x_b \end{aligned}$$

so that:

$$\begin{aligned} (-5-\lambda)x_a + \quad 2x_b \quad &= 0 \\ 2x_a \quad + (-2-\lambda)x_b &= 0 \end{aligned}$$

or: $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

which is homogeneous. By Cramer's rule it has solution $\mathbf{x} \neq \mathbf{0}$ iff its coefficient determinant is zero.

Example 1

Determination of eigenvalues and eigenvectors

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solution: eigenvalues.

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda)-4 = \lambda^2+7\lambda+6 = 0$$

We call $D(\lambda)$ the **characteristic determinant** or, if expanded, the **characteristic polynomial**, and $D(\lambda)=0$ the **characteristic equation** of **A**.

The solutions of this quadratic equation are $\lambda_1=-1$ and $\lambda_2=-6$. These are the eigenvalues of **A**

Example 1

Determination of eigenvalues and eigenvectors

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solution: eigenvector of A corresponding to $\boldsymbol{\lambda_1}$.

This vector is found by setting $\lambda=\lambda_1=-1$ in the original equations

$$\begin{aligned} (-5-\lambda)x_a + \quad 2x_b &= 0 \\ 2x_a + (-2-\lambda)x_b &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} -4x_a + 2x_b &= 0 \\ 2x_a - \quad x_b &= 0 \end{aligned}$$

i.e. $x_b=2x_a$ where x_a is chosen arbitrarily. Let $x_a=1$ then $x_b=2$ and an eigenvector corresponding to $\lambda_1=-1$ is

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{Check:} \quad \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda\begin{bmatrix} -1 \\ -2 \end{bmatrix} = -1\mathbf{x_1} = \lambda_1\mathbf{x_1}$$

Example 1

Determination of eigenvalues and eigenvectors

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solution: eigenvector of A corresponding to λ_2.

Set $\lambda=\lambda_2=-6$ in the original equations

$$\begin{aligned} (-5-\lambda)x_a + 2x_b &= 0 \\ 2x_a + (-2-\lambda)x_b &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_a + 2x_b &= 0 \\ 2x_a + 4x_b &= 0 \end{aligned}$$

i.e. $x_b=-x_a/2$ where x_a is chosen arbitrarily. Let $x_a=2$ then $x_b=-1$ and an eigenvector corresponding to $\lambda_2=-6$ is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$