ERG 2012B
Advanced Engineering Mathematics II

Part II: Linear Algebra

Lecture #13
Linear Systems and Determinant

Linear Dependence

Theorem 4: Linear Dependence and Independence

p vectors (with n components each) are linearly independent if the matrix of these row vectors has rank p.

They are linearly dependent if the rank is less than p

Theorem 5: p vectors with n components each and $n < p$ are always linearly dependent.

Proof: Since each of these p vectors has n components, the corresponding matrix **A** is p x n. The number of columns is n therefore rank $\mathbf{A} \leq n < p$. Hence linearly dependent.

Theorem 6: The vector space $\Re^{\mathbf{n}}$ consisting of all vectors with n components has dimension n

General Properties of Solutions

Fundamental Theorem for Linear Systems

(a) A linear system of *m* equations in *n* unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad (1)$$

has solutions iff the coefficient matrix and the augmented matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{12} \\ & \cdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \text{ and } \widetilde{\mathbf{A}} = \begin{bmatrix} a_{11} & \cdots & a_{12} & b_1 \\ & \cdots & & \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix}$$

have the same rank

General Properties of Solutions

Fundamental Theorem for Linear Systems

(b) If this rank r equals n then system has precisely one solution

(c) If $r < n$ the system has infinitely many solutions, all of which are obtained by determining r suitable unknowns in terms of the remaining n-r unknowns, to which arbitrary values can be given

(d) If solutions exist, they can all be obtained by the Gauss elimination method

General Properties of Solutions

Fundamental Theorem for Linear Systems

Proof: (a) solution iff rank **A** = rank **Ã**

If **A** has a solution **x** then **Ax=b** or in column vectors:

$$\mathbf{c_1}x_1 + \mathbf{c_2}x_2 + \cdots + \mathbf{c_n}x_n = \mathbf{b}$$

Since **Ã** is obtained by attaching to **A** the additional column **b** theorem 1 says that rank **Ã** equals rank **A** or rank **A**+1.

And **b** is a linear combination of the column vectors $\mathbf{c_n}$ (above)

Hence rank **Ã** cannot exceed rank **A** so rank **Ã** = rank **A**

Similarly if rank **Ã** = rank **A** then **b** must be a linear combination of the column vectors of **A**

$$\mathbf{c_1}\alpha_1 + \mathbf{c_2}\alpha_2 + \cdots + \mathbf{c_n}\alpha_n = \mathbf{b}$$

Which implies that $x_1=\alpha_1, \ldots\ldots x_n=\alpha_n$ hence a solution

General Properties of Solutions

Fundamental Theorem for Linear Systems

Proof: (b) if rank $\mathbf{A} = r = n$ there is precisely one solution

If rank $\mathbf{A} = r = n$ then the set C = {$\mathbf{c_1}, \ldots, \mathbf{c_n}$} is linearly independent - theorem 1.

The representation of **b** i.e. $\mathbf{c_1}x_1 + \mathbf{c_2}x_2 + \ldots + \mathbf{c_n}x_n = \mathbf{b}$ must be unique because otherwise:

$$\mathbf{c_1}x_1 + \mathbf{c_2}x_2 + \cdots + \mathbf{c_n}x_n = \mathbf{c_1}\tilde{x}_1 + \mathbf{c_2}\tilde{x}_2 + \cdots + \mathbf{c_n}\tilde{x}_n$$

would imply

$$\mathbf{c_1}(x_1 - \tilde{x}_1) + \cdots + \mathbf{c_n}(x_n - \tilde{x}_n) = 0$$

so that $x_1 - \tilde{x}_1 = 0, \cdots, x_n - \tilde{x}_n = 0$

Hence the solution is the same one.

General Properties of Solutions

Fundamental Theorem for Linear Systems

Proof: (c) if rank **A** = $r < n$ theorem 1 says there is a linearly independent set K of r column vectors of **A** such that the other n-r column vectors of **A** are linear combinations of those vectors...........

Example 1

$$\begin{cases} 3x_1 + 2x_2 + 2x_3 - 5x_4 = 8 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7 \\ 1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1 \end{cases}$$

$$\Rightarrow \tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

By Gauss elimination $\tilde{\mathbf{A}}$ is row equivalent to;

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \text{rank } \mathbf{A} = \text{rank } \tilde{\mathbf{A}} = 2 < n = 4$

We can choose x_3 & x_4 arbitrarily

Example 2

$$\begin{cases} -x_1 + x_2 + 2x_3 = 2 \\ 3x_1 - x_2 + x_3 = 6 \\ -x_1 + 3x_2 + 4x_3 = 4 \end{cases}$$

$$\Rightarrow \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{array}\right]$$

By Gauss elimination $\tilde{\mathbf{A}}$ is row equivalent to;

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

$\therefore$ rank A = rank $\tilde{\mathbf{A}}$ = 3 = n=3

1 unique solution

Example 3

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + x_2 + x_3 = 0 \\ 6x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

$$\Rightarrow \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

By Gauss elimination $\tilde{\mathbf{A}}$ is row equivalent to;

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

$\therefore \text{rank } \mathbf{A} = 2 < \text{rank } \tilde{\mathbf{A}} = 3$

therefore no solution

The Homogeneous System

The system **Ax** = **b** is called **homogenous** if **b**=0. Otherwise it is called **nonhomogenous**

Theorem 2 (Homogeneous System)

A homogeneous linear system **Ax** = **0** or

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

always has the trivial solution **x**=**0**. Nontrivial solutions exist iff rank **A** = $r < n$, these solutions, together with **x**=**0** form a vector space of dimension n-r.

Proof: (see text book) but fairly obvious from definition of vector space and general properties.

The Homogeneous System

- The vector space of all solutions is called the **null space** of **A** because if we multiply any **x** in this null space by **A** we get 0

- The dimension of the null space is called the **nullity** of **A**

- Theorem 2 states that

  $$\text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = n$$

  where n is the number of unknowns

- If rank **A** = n then nullity **A** = 0 - i.e. trivial solution only

- If rank **A** = $r < n$ then nullity **A** = n-$r > 0$

Theorems

Theorem 3: Systems with fewer equations than unknowns A homogeneous system of linear equations with fewer equations than unknowns always has non-trivial solutions

Proof: $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{A}$ is n x m (m equations, n unknowns)

Since rank $\mathbf{A} \leq m$ and $m < n$ then rank $\mathbf{A} < n$

If rank $\mathbf{A} = r < n$ then nullity $\mathbf{A} = n\text{-}r > 0$ and so has non-trivial roots

Theorem 4: Nonhomogeneous System. If a nonhomogeneous linear system $\mathbf{Ax} = \mathbf{b}$ ($\neq 0$) has solutions then all these solutions are of the form: $\mathbf{x} = \mathbf{x_0}+\mathbf{x_h}$ where $\mathbf{x_0}$ is any fixed solution and $\mathbf{x_h}$ runs through all the solutions of the corresponding homogeneous system $\mathbf{Ax}=\mathbf{0}$

Proof: Let $\mathbf{x}$ be any solution and $\mathbf{x_0}$ any chosen one. Then $\mathbf{Ax}=\mathbf{b}$ and $\mathbf{Ax_0}=\mathbf{b}$ and so $\mathbf{A(x\text{-}x_0)}=\mathbf{Ax\text{-}Ax_0}=\mathbf{0}$. So that $\mathbf{x\text{-}x_0}$ is a solution of the homogeneous system and $\mathbf{x\text{-}x_0}=\mathbf{x_h}$ in general

Inverse of a Matrix

The **inverse** of an n x n matrix $\mathbf{A}=[a_{jk}]$ is denoted $\mathbf{A}^{-1}$ and is an n x n matrix such that:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where **I** is the n x n unit matrix

- If **A** has an inverse, then **A** is called a **nonsingular matrix**
- If **A** has no inverse, then **A** is called a **singular matrix**

If **A** has an inverse, the inverse is unique

Proof: If both **B** and **C** are inverses of **A** then **AB**=**I** and **CA=I** so that $\mathbf{B}=\mathbf{IB}=(\mathbf{CA})\mathbf{B}=\mathbf{C}(\mathbf{AB})=\mathbf{CI}=\mathbf{C}$

Existence of the Inverse

Theorem 1: The **inverse** of an n x n matrix $\mathbf{A}$ exists iff rank $\mathbf{A}$ = n. Hence $\mathbf{A}$ is nonsingular if rank $\mathbf{A}$=n and is singular if rank $\mathbf{A} < n$

Proof: Consider the system $\mathbf{Ax}$=$\mathbf{B}$ with the given matrix $\mathbf{A}$ as the coefficient matrix. If the inverse exists then

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

This shows that $\mathbf{Ax}$=$\mathbf{b}$ has a unique solution $\mathbf{x}$, so that $\mathbf{A}$ must have rank n

Conversely, if rank $\mathbf{A}$=n then $\mathbf{Ax}$=$\mathbf{b}$ has a unique solution $\mathbf{x}$ for any $\mathbf{b}$ and the back substitution following Gauss elimination shows that its components x_j are linear combinations of those of $\mathbf{b}$ so we can write $\mathbf{x} = \mathbf{Bb}$

So that $\mathbf{Ax} = \mathbf{A(Bb)} = \mathbf{(AB)b} = \mathbf{b}$ and so $\mathbf{AB} = \mathbf{I}$ **or** $\mathbf{B}$=$\mathbf{A}^{-1}$

Determination of the Inverse

For practically determining the inverse $\mathbf{A}^{-1}$ of a non-singular *n*x*n* matrix $\mathbf{A}$ we can use a variant of Gauss elimination - **Gauss-Jordan elimination**

Using $\mathbf{A}$ we form the *n* systems $\mathbf{Ax_1}=\mathbf{e_1},\ldots,\mathbf{Ax_n}=\mathbf{e_n}$ where $\mathbf{e_j}$ is a column vector with the *j*th component 1 and all the others 0

Introducing the *n*x*n* matrices $\mathbf{X}=[\mathbf{x_1}\ldots\mathbf{x_n}]$ and $\mathbf{I} = [\mathbf{e_1}\ldots\mathbf{e_n}]$ we can combine the *n* systems into the matrix equation $\mathbf{AX}=\mathbf{I}$ and the *n* augmented matrices $[\mathbf{A}\ \mathbf{e_1}],\ldots,[\mathbf{A}\ \mathbf{e_n}]$ into a single augmented matrix $\tilde{\mathbf{A}} = [\mathbf{A}\ \mathbf{I}]$

Now $\mathbf{AX}=\mathbf{I}$ implies $\mathbf{X}=\mathbf{A}^{-1}\mathbf{I}=\mathbf{A}^{-1}$ and to solve $\mathbf{AX}=\mathbf{I}$ for $\mathbf{X}$ we can use Gauss elimination to $\tilde{\mathbf{A}}$ to get $[\mathbf{U}\ \mathbf{H}]$ where $\mathbf{U}$ is upper triangular

The Gauss-Jordan elimination operates on $[\mathbf{U}\ \mathbf{H}]$ by eliminating the entries in $\mathbf{U}$ above the diagonal giving $[\mathbf{I}\ \mathbf{K}]$ the augmented matrix of $\mathbf{IX}=\mathbf{A}^{-1}$ Thus $\mathbf{K}=\mathbf{A}^{-1}$

Gauss-Jordan Elimination

Example: Find the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Solution: Gauss Elimination gives:

$$[\mathbf{A} \quad \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \begin{array}{l} \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \text{row2+3row1} \\ \text{row3 - row1} \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \text{row3 - row2} \end{array}$$

This is [**U** **H**] as produced by Gauss elimination

The additional Gauss-Jordan steps reduce **U** to **I** - next page

Gauss-Jordan Elimination

$$
[\mathbf{U}\ \ \mathbf{H}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{-row1} \\ \text{0.5 row2} \\ \text{-0.2 row3} \end{array}
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{row1 + 2row3} \\ \text{row2 - 3.5row3} \\ \end{array}
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{row1 + row2} \\ \\ \end{array}
$$

The last three columns give $\mathbf{A}^{-1}$ - check for yourself.......