ERG 2012B
Advanced Engineering Mathematics II

Part II: Linear Algebra

Lecture #12
Matrices and Linear Equations

Motivation of Matrix X by L.T.

- Consider we have two quantities y_1 and y_2 which are linearly related to two other variables x_1 and x_2, given by:

$$y_1 = a_{11}.x_1 + a_{12}.x_2$$

$$y_2 = a_{21}.x_1 + a_{22}.x_2$$

- We say that $\bar{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are related by a linear transformation characterized by the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

In Matrix form $\bar{\mathbf{y}} = \mathbf{A}.\bar{\mathbf{x}}$

Motivation of Matrix X by L.T.

• Now suppose the variables x_1 and x_2 are further dependent linearly on another pair of variables, w_1 and w_2, given by

$$x_1 = b_{11}.w_1 + b_{12}.w_2$$

$$x_2 = b_{21}.x_1 + b_{22}.w_2$$

Or in Matrix form $\overline{\mathbf{x}} = \mathbf{B}.\overline{\mathbf{w}}$ where $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

The relation between $\overline{\mathbf{y}}$ and $\overline{\mathbf{w}}$ can then be obtained by direct substitution. It is straightforward to verify that the results is given by:

$$\overline{\mathbf{y}} = \mathbf{A}.\overline{\mathbf{x}} = \mathbf{A}.(\mathbf{B}.\overline{\mathbf{w}}) = (\mathbf{A}.\mathbf{B}).\overline{\mathbf{w}} = \mathbf{C}.\overline{\mathbf{w}}$$

where $\mathbf{C} = \mathbf{A}.\mathbf{B}$ is as defined by matrix multiplication

•For higher dimensions the idea and the result is exactly the same If there are *m* variables y_1, ...y_m; *n* variables x_1,...x_n and *p* variables w_1,...w_p, then **A** is *m* x *n*, **B** is *n* x *p* and **C** is *m* x *p*

Linear Systems of Equations

- A **linear system** of *m* equations in *n* unknowns $x_1 \dots x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}.x_1 + \cdots + a_{1n}.x_n &= b_1 \\ a_{21}.x_1 + \cdots + a_{2n}.x_n &= b_2 \\ &\cdots\cdots \\ a_{m1}.x_1 + \cdots + a_{mn}.x_n &= b_m \end{aligned} \qquad (1)$$

- The a_{jk} are given numbers, the **coefficients** of the system
- The b_i are also given numbers
  - if all $b_i=0$ then (1) is called a **homogeneous system**
  - if at least one $b_i \neq 0$ then (1) is **nonhomogeneous**
- A solution of (1) is a set of numbers $x_1,\dots x_n$ that satisfy all *m* equations. A **solution vector** of (1) is a vector whose components constitute a solution of (1)
- If the system is homogeneous, it has at least the **trivial solution** $\bar{\mathbf{x}} = 0$

Linear Systems of Equations

- A **linear system** of *m* equations in *n* unknowns $x_1 \ldots x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}.x_1 + \cdots + a_{1n}.x_n &= b_1 \\ a_{21}.x_1 + \cdots + a_{2n}.x_n &= b_2 \\ \cdots\cdots \\ a_{m1}.x_1 + \cdots + a_{mn}.x_n &= b_m \end{aligned} \qquad (1)$$

The system (1) can be represented in matrix form as:

$$\mathbf{A}.\overline{\mathbf{x}} = \overline{\mathbf{b}}$$

with an *n* x *m* **coefficient matrix** $\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

an *n* component column vector $\overline{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

and an *m* component column vector $\overline{\mathbf{b}} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

Linear Systems of Equations

- A **linear system** of *m* equations in *n* unknowns $x_1 \ldots x_n$ is a set of equations of the form

$$
\begin{aligned}
a_{11}.x_1 + \cdots + a_{1n}.x_n &= b_1 \\
a_{21}.x_1 + \cdots + a_{2n}.x_n &= b_2 \\
&\cdots\cdots \\
a_{m1}.x_1 + \cdots + a_{mn}.x_n &= b_m
\end{aligned}
\qquad (1)
$$

The matrix

$$
\tilde{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}
$$

is called the **augmented matrix** of the system. It determines the system completely.

Geometric Interpretation

Existence of Solutions

If $m=n=2$, we have two equations in two unknowns x_1, x_2

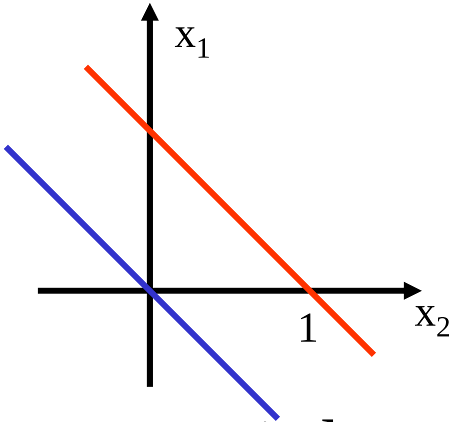
$$a_{11}.x_1 + a_{12}.x_2 = b_1$$
$$a_{21}.x_1 + a_{22}.x_2 = b_2$$

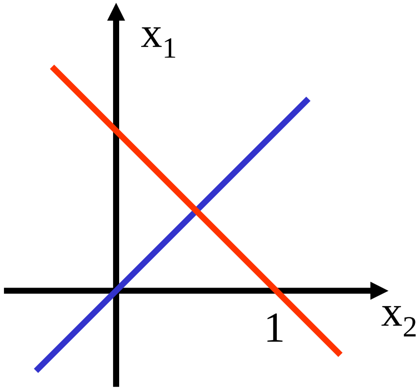
If we interpret x_1, x_2 as coordinates in the x_1x_2-plane then each equation represents a straight line and the point P is a solution iff it lies on both lines. There are 3 possible cases:

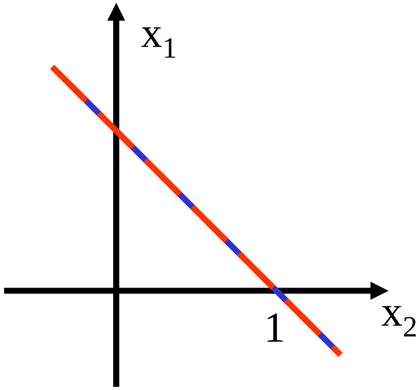
a) No solution if lines are parallel
e.g $x_1+x_2=1$; $x_1+x_2=0$

b) A single solution if they intersect
e.g $x_1+x_2=1$; $-x_1+x_2=0$

c) infinitely many if they coincide
e.g $x_1+x_2=1$; $2x_1+2x_2=2$







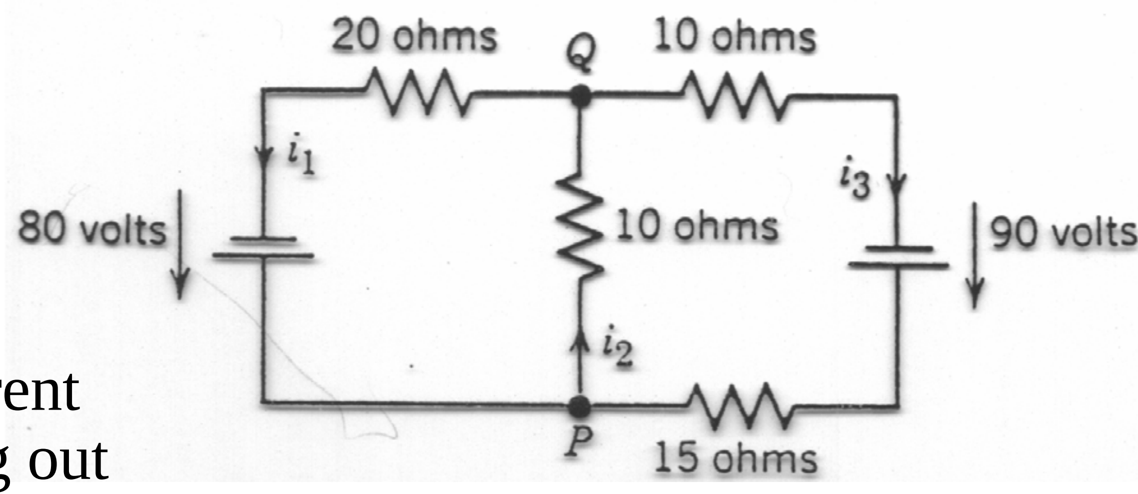
If the system is homogeneous, case a) cannot happen as both lines must pass through the origin.

Gauss Elimination

The linear system:

$$i_1 - i_2 + i_3 = 0$$
$$-i_1 + i_2 - i_3 = 0$$
$$10i_2 + 25i_3 = 90$$
$$20i_1 + 10i_2 = 80$$

is derived from the circuit for the unknown currents



The equations come from applying Kirchoff's Laws

Kirchoff's Current Law: At any point of a circuit, the current flowing in equals that flowing out

Kirchoff's Voltage Law: In any closed loop the sum of all voltage drops equals the applied emf

Node **P** gives the first equation; node **Q** gives the second. The right loop the third and the left loop the fourth.

Gauss Elimination

The linear system:

pivot

$$
\begin{aligned}
i_1 - i_2 + i_3 &= 0 \\
-i_1 + i_2 - i_3 &= 0 \\
10i_2 + 25i_3 &= 90 \\
20i_1 + 10i_2 &= 80
\end{aligned}
$$

eliminate

The Augmented Matrix is:

$$
\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}
$$

This system of equations is very simple to solve. But using a systematic method - Gauss elimination - will work in general even for very large more complex systems.

The aim is to reduce the system to a **triangular form** from which we can obtain the values by **back substitution**

First Step: Elimination of i_1

We use the first equation (**pivot equation**) and the first term (**pivot**) to eliminate i_1 in the other equations

subtract -1 times the pivot equation from the second equation

subtract 20 times the pivot equation from the fourth equation

Gauss Elimination

The linear system:

$$\begin{aligned} i_1 - i_2 + i_3 &= 0 \\ 0 &= 0 \\ 10 i_2 + 25 i_3 &= 90 \\ 30 i_2 - 20 i_3 &= 80 \end{aligned}$$

The Augmented Matrix is:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix} \begin{matrix} \\ \text{row2+row1} \\ \\ \text{row4-20row1} \end{matrix}$$

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Gauss Elimination

The linear system:

$$
\begin{aligned}
i_1 - i_2 + i_3 &= 0 \\
0 &= 0 \\
10 i_2 + 25 i_3 &= 90 \\
30 i_2 - 20 i_3 &= 80
\end{aligned}
$$

The Augmented Matrix is:

$$
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 10 & 25 & 90 \\
0 & 30 & -20 & 80
\end{bmatrix}
$$

This system of equations is very simple to solve. But using a systematic method - Gauss elimination - will work in general even for very large more complex systems.

The aim is to reduce the system to a **triangular form** from which we can obtain the values by **back substitution**

Second Step: Elimination of i_2

The first equation now remains untouched and we use the new second equation as the next pivot equation. But since it contains no term in i_2 we change the order of the equations to get a non-zero pivot.

Gauss Elimination

The linear system:

$$
\begin{aligned}
i_1 - i_2 + i_3 &= 0 \\
\text{pivot} \quad 10i_2 + 25i_3 &= 90 \\
\text{eliminate} \quad 30i_2 - 20i_3 &= 80 \\
0 &= 0
\end{aligned}
$$

The Augmented Matrix is:

$$
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 10 & 25 & 90 \\
0 & 30 & -20 & 80 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

This system of equations is very simple to solve. But using a systematic method - Gauss elimination - will work in general even for very large more complex systems.

The aim is to reduce the system to a **triangular form** from which we can obtain the values by **back substitution**

Second Step: Elimination of i_2

We can now use equation 2 as a pivot and eliminate i_2 from the lines 3 and 4. We subtract 3 times the pivot equation from the third equation

Gauss Elimination

The linear system:

$$
\begin{aligned}
i_1 - i_2 + i_3 &= 0 \\
10i_2 + 25i_3 &= 90 \\
-95i_3 &= -190 \\
0 &= 0
\end{aligned}
$$

The Augmented Matrix is:

$$
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 10 & 25 & 90 \\
0 & 0 & -95 & -190 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{row3-2row2}
$$

This system of equations is very simple to solve. But using a systematic method - Gauss elimination - will work in general even for very large more complex systems.

The aim is to reduce the system to a **triangular form** from which we can obtain the values by **back substitution**

Second Step: Elimination of i_2

We can now use equation 2 as a pivot and eliminate i_2 from the lines 3 and 4. We subtract 3 times the pivot equation from the third equation

Gauss Elimination

The linear system:

$$\begin{aligned} i_1 - i_2 + i_3 &= 0 \\ 10i_2 + 25i_3 &= 90 \\ -95i_3 &= -190 \\ 0 &= 0 \end{aligned}$$

The Augmented Matrix is:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Working backward from the last to the first equation of this triangular system we can now readily find i_3, then i_2 and then i_1:

$$\begin{aligned} i_1 - i_2 + i_3 &= 0 \\ 10i_2 + 25i_3 &= 90 \\ -95i_3 &= -190 \\ 0 &= 0 \end{aligned}$$

$i_1 = i_2 - i_3 = 2$ [amperes]

$i_2 = (90-25i_3)/10 = 4$ [amperes]

$i_3 = 2$ [amperes]

This solution is unique for this system

Gauss Elimination

- A system is called **overdetermined** if it has more equations than unknowns, as in the example.
- A system is called **determined** if $m = n$, as in the first example
- A system is called **underdetermined** if it has fewer equations than unknowns.
- A system may have:
  - one solution or
  - more than one solution or
  - no solutions at all

We will discuss the details of this later.

Gauss Elimination

Gauss elimination for an underdetermined system

Solve the linear system of three equations in four unknowns:

$$3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0$$
$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$
$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$$

First Step: Elimination of x_1

We eliminate x_1 in the other equations by subtracting

0.6/3.0 = 0.2 times the pivot equation from the second equation

1.2/3.0 = 0.4 times the pivot equation from the third equation

Gauss Elimination

Gauss elimination for an underdetermined system

Solve the linear system of three equations in four unknowns:

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ -1.1x_2 - 1.1x_3 + 4.4x_4 &= -1.1 \end{aligned} \qquad \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{bmatrix}$$

Second Step: Elimination of x_2

We eliminate x_2 in the third equation by subtracting

-1.1/1.1 = -1 times the pivot equation from the third equation

Gauss Elimination

Gauss elimination for an underdetermined system

Solve the linear system of three equations in four unknowns:

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0 \end{aligned}$$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Back Substitution: From the second equation $x_2=1-x_3+4x_4$. From this and the first equation $x_1=2-x_4$ Since x_3 and x_4 remain arbitrary, we have infinitely many solutions; if we chose a value of x_3 and x_4 then the x_1 and x_2 can be uniquely determined.

Example - Unique Solution

Solve the linear system

$$\begin{aligned} -x_1 + x_2 + 2x_3 &= 2 \\ 3x_1 - x_2 + x_3 &= 6 \\ -x_1 + 3x_2 + 4x_3 &= 4 \end{aligned} \qquad \begin{bmatrix} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{bmatrix}$$

First step eliminate x_1 from 2nd and 3rd equations gives:

$$\begin{aligned} -x_1 + x_2 + 2x_3 &= 2 \\ 2x_2 + 7x_3 &= 12 \\ 2x_2 + 2x_3 &= 2 \end{aligned} \qquad \begin{bmatrix} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{bmatrix} \quad \begin{matrix} \\ \text{row 2 + 3 row 1} \\ \text{row 3 – row 1} \end{matrix}$$

Second step eliminate x_2 from 3rd equation gives:

$$\begin{aligned} -x_1 + x_2 + 2x_3 &= 2 \\ 2x_2 + 7x_3 &= 12 \\ -5x_3 &= -10 \end{aligned} \qquad \begin{bmatrix} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{bmatrix} \quad \begin{matrix} \\ \\ \text{row 3 – row 2} \end{matrix}$$

Back Substitution: starting at the last equation, we obtain successively $x_3=2$, $x_2=-1$, $x_1=1$.

Example - No Solution

Solve the linear system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 4x_3 &= 6 \end{aligned} \qquad \begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

First step eliminate x_1 from 2nd and 3rd equations gives:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ -2x_2 + 2x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\tfrac{1}{3} & \tfrac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

row 2 - 2/3 row 1

row 3 – 2 row 1

Second step eliminate x_2 from 3rd equation gives:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ 0 &= 12 \end{aligned} \qquad \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\tfrac{1}{3} & \tfrac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

row 3 - 6 row 2

The resulting contradiction demonstrates that the system has no solution

Echelon Form

The form of the system and of the matrix in the last step of the Gauss elimination is called the **echelon form**. Thus in the last example the echelon forms of the coefficient and augmented matrices are:

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

At the end of the Gauss elimination the system has the form:

$$\begin{aligned} a_{11}x_1 + a_{22}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ c_{22}x_2 + \cdots\cdots + c_{1n}x_n &= b_2 \\ \cdots\cdots & \\ k_{rr}x_r + \cdots + k_{rn}x_n &= b_r \\ 0 &= b_{r+1} \\ &\vdots \\ 0 &= b_m \end{aligned}$$

where $r \leq m$ ($a_{11} \neq 0$, $c_{22} \neq 0$,...$k_{rr} \neq 0$)

From this we see that:

a) No solution if r<m and one of the numbers b_{r+1}...b_m is not zero

b) Precisely one solution if r=n and b_{r+1}...b_m, are zero

c) Infinitely many solutions if r<n and b_{r+1}...b_m are zero

Row Operations for Matrices

Gauss elimination consists of the use three operations on a linear system of equations:

Elementary operations for equations

- Interchange of two equations
- Multiplication of an equation by a nonzero constant
- Addition of a constant multiple of one equation to another

Elementary row operations for matrices

- Interchange of two rows
- Multiplication of a row by a nonzero constant
- Addition of a constant multiple of one row to another

We call a linear system S_1 **row equivalent** to a linear system S_2 if S_1 can be obtained from S_2 by these elementary row operations.

Row-Equivalent Systems

Theorem 1

Row-equivalent linear systems have the same set of solutions

Proof

The interchange of two equations does not alter the solution set

Neither does the multiplication of an equation by a nonzero constant c

Addition of an equation E_1 to an equation E_2 similarly does not alter the solution set

Linear Independence

- Consider a **linear combination of** any set of *m* vectors:

$$\sum_{i=1}^{m} \mathrm{c}_i \bar{\mathbf{a}}_{\mathbf{i}} = \mathrm{c}_1 \bar{\mathbf{a}}_1 + \mathrm{c}_2 \bar{\mathbf{a}}_2 + \cdots + \mathrm{c}_m \bar{\mathbf{a}}_m$$

where the c_i's are **scalars**.

If $$\sum_{i=1}^{m} \mathrm{c}_i \bar{\mathbf{a}}_{\mathbf{i}} = 0 \Rightarrow \text{all } \mathrm{c}_i = 0$$

then the *m* vectors are said to be **linearly independent.**.

Otherwise, if the sum is zero and there exists a set of c_i not all zero, then the *m* vectors are said to be **linearly dependent** and we can express at least one of the equations as a linear combination of the others

Example 1

$$\bar{\mathbf{a}}_1 = [3 \quad 0 \quad 2 \quad 2]$$

$$\bar{\mathbf{a}}_2 = [-6 \quad 42 \quad 24 \quad 54]$$

$$\bar{\mathbf{a}}_3 = [21 \quad -21 \quad 0 \quad -15]$$

$\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3$ are linearly dependent since

$$6\bar{\mathbf{a}}_1 - \tfrac{1}{2}\bar{\mathbf{a}}_2 - \bar{\mathbf{a}}_3 = 0$$

but $\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2$ are linearly independent

$$\because \text{If} \quad c_1\bar{\mathbf{a}}_1 + c_2\bar{\mathbf{a}}_2 = 0$$

$$\Rightarrow [3c_1 - 6c_2 \quad 42c_2 \quad 2c_1 + 24c_2 \quad 2c_1 + 54c_2] = 0$$

$$\Rightarrow c_2 = 0 \text{ and } c_1 = 0$$

Vector Space

Definition: Given *m* vectors with *n* components each.

Let V be the set of all linear combinations of these vectors.

- The set V with two operations, addition and scalar multiplication forms a **vector space**
- The maximum number of linearly independent vectors in V is called the **dimension** of V is denoted by **dim V**
- If the given *m* vectors are
  - linearly independent then dim V = *m*
  - linearly dependent then dim V $<$ *m*
- A linearly independent set in V consisting of a maximum possible number of vectors in V is called a **basis** for V
- The number of vectors of a **basis** for V equals dim V

Example 2

The span of the three vectors in the previous example

$$\bar{\mathbf{a}}_1 = \begin{bmatrix} 3 & 0 & 2 & 2 \end{bmatrix}$$
$$\bar{\mathbf{a}}_2 = \begin{bmatrix} -6 & 42 & 24 & 54 \end{bmatrix}$$
$$\bar{\mathbf{a}}_3 = \begin{bmatrix} 21 & -21 & 0 & -15 \end{bmatrix}$$

is a vector space of dimension 2

A basis of this set is:

$$\left\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2\right\} \text{or} \left\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_3\right\} \text{etc..}$$

Vector Space

- The **real *n*-dimensional vector space $\Re^{\mathbf{n}}$** is the space of all vectors with *n* real numbers as components and real numbers as scalars.
- Each such vector is an ordered *n*-tuple of real numbers.

Example

For $n = 3$ we get $\Re^3$ consisting of ordered triples (vectors in 3-D)

For $n = 2$ we get $\Re^2$ consisting of ordered pairs (vectors in a plane)

Rank of a Matrix

- The maximum number of linearly independent row vectors of a matrix **A** = $[a_{jk}]$ is called the **rank of A** and is denoted

rank **A**

- rank **A** = 0 ⇔ **A** = **0**

Example 3: the matrix

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

has rank 2

– we showed in example 1 that the first 2 row vectors are linearly independent but all three row vectors are linearly dependent.

Rank in Terms of Column Vectors

Theorem 1: The rank of a matrix **A** equals the maximum number of linearly independent column vectors of **A**.

Hence **A** and $\mathbf{A}^T$ have the same rank

Definition:

- The span of the row vectors of a matrix **A** is called the **row space of A** and
- the span of the column vectors the **column space of A**

Theorem 2: The row and column space of a matrix **A** have the same dimension, equal to rank **A**

Rank in Terms of Column Vectors

Proof of Theorem 1: Let **A**=[a_{jk}] and let rank **A** = r. Then **A** has a linearly independent set of row vectors **V** and all *row* vectors of **A** are linear combinations of the independent ones

These are vector equations. Each is equivalent to *n* equations which can be expanded out to:

$$\begin{aligned}
\bar{\mathbf{a}}_1 &= c_{11}\bar{\mathbf{v}}_1 + c_{12}\bar{\mathbf{v}}_2 + \cdots\cdots + c_{1r}\bar{\mathbf{v}}_r \\
\bar{\mathbf{a}}_2 &= c_{21}\bar{\mathbf{v}}_1 + c_{22}\bar{\mathbf{v}}_2 + \cdots\cdots + c_{2r}\bar{\mathbf{v}}_r \\
&\vdots \\
\bar{\mathbf{a}}_m &= c_{m1}\bar{\mathbf{v}}_1 + c_{m2}\bar{\mathbf{v}}_2 + \cdots\cdots + c_{mr}\bar{\mathbf{v}}_r
\end{aligned}$$

$$\begin{aligned}
a_{1k} &= c_{11}v_{1k} + c_{12}v_{2k} + \cdots\cdots + c_{1r}v_{rk} \\
a_{2k} &= c_{21}v_{1k} + c_{22}v_{2k} + \cdots\cdots + c_{2r}v_{rk} \\
&\vdots \\
a_{mk} &= c_{m1}v_{1k} + c_{m2}v_{2k} + \cdots\cdots + c_{mr}v_{rk}
\end{aligned}$$

where k=1,.....,n

These can be written in columns as a set of vectors (next page)

Rank in Terms of Column Vectors

Proof of Theorem 1 (cont'd)

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} = v_{1k}\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} + v_{2k}\begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{bmatrix} + \cdots + v_{rk}\begin{bmatrix} c_{1r} \\ c_{2r} \\ \cdots \\ c_{mr} \end{bmatrix}$$

The vector on the left is the k^{th} *column* vector of **A**. Each column vector of **A** is a linear combination of the r vectors on the right. Hence the maximum number of linearly independent column vectors of **A**, r_c, cannot exceed r – i.e. $r_c \leq r$

If we apply the same argument to $\mathbf{A}^T$ we get the maximum number of independent row vectors of **A**, r (the rank), cannot exceed the r_c i.e. $r \leq r_c$

Therefore $r_c = r$

Example 4

It is easily verified that for the matrix **A** in example 3:

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

the first two column vectors are linearly independent and

$$\begin{bmatrix} 2 \\ 24 \\ 0 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 3 \\ -6 \\ 21 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 0 \\ 42 \\ -21 \end{bmatrix} \quad \text{and} \begin{bmatrix} 2 \\ 54 \\ -15 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 3 \\ -6 \\ 21 \end{bmatrix} + \frac{29}{21}\begin{bmatrix} 0 \\ 42 \\ -21 \end{bmatrix}$$

Since rank **A** = 2 the maximum number of linearly independent column matrices is also 2

Invariance of Rank

Invariance of Rank under Elementary Row Operations

Elementary row operations do not alter the rank of a matrix **A**

Theorem 3: *Row equivalent matrices have the same rank*

- A practical method to determine the rank of a matrix:
  - Reduce A to echelon form by Gauss elimination
  - From the echelon form the rank can be recognized easily

Example 5

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly rank **A** = 2