ERG 2012B
Advanced Engineering Mathematics II

Part II: Linear Algebra

Lecture #11
Matrices and Linear Equations

Linear Algebra

- **Matrix**: a rectangular array of numbers (or functions), called entries or **elements** of the matrix, enclosed in brackets.

e.g. $\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ ← rows: horizontal lines

↑ columns: vertical lines

the matrix **M** has 3 columns and 2 rows

- An ***m* x *n* matrix** – *m* rows and *n* columns
- **Double Subscript notation** for matrix entries
  - 1st subscript denotes the row
  - 2nd subscript denotes column

$$\mathbf{A}_{\mathbf{m \times n}} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots\cdots & a_{mn} \end{bmatrix}_{\mathbf{m \times n}}$$

Matrices and Vectors

- If an m x n matrix has m=n, it is called an ***n* x *n* square matrix**
  - The entries a_{11}, a_{22},.....a_{nn} are called the **main** or **principal diagonal** of the matrix A
- A **submatrix** of and m x n matrix **A** is a matrix obtained by omitting some rows or columns (or both) from **A**
- For convenience, this includes **A** itself (i.e. omitting no rows or columns)

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots & a_{2n} \\ \dots & \dots & \dots\dots\dots & \dots \\ a_{m1} & a_{m2} & \dots\dots\dots & a_{mn} \end{bmatrix}$$

- **Vectors**: column/row vectors are a single column/row matrix

The entries of a vector are called its **components**

$$\bar{\mathbf{a}} = [a_1 \; a_2 \; \dots\dots \; a_n]$$

row vector

$$\bar{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

column vector

Transposition of a Matrix

- The **transpose** $\mathbf{A}^T$ of an m x n matrix $\mathbf{A}=[a_{jk}]$ is the n x m matrix that has the first rows of **A** as its first column, the second row of **A** as its second column and so on.

The transpose of **A** is:

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \quad \mathbf{A}^T = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots\cdots & a_{m1} \\ a_{12} & a_{22} & \cdots\cdots & a_{m2} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{1n} & a_{2n} & \cdots\cdots & a_{nm} \end{bmatrix}$$

- **Example**

$$\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

- The transpose of a column vector is a row vector and vice versa

Symmetric Matrices

- A **Symmetric Matrix** is a square matrix whose transpose is the same as the original matrix – i.e. $\mathbf{A}^T = \mathbf{A}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

- A **Skew-symmetric Matrix** is a square matrix whose transpose is the same as the negative of the original matrix – i.e. $\mathbf{A}^T = -\mathbf{A}$

- Two Matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A}=\mathbf{B}$, if and only if they have the same size and the corresponding entries are all equal; i.e. $a_{jk} = b_{jk}$ for every j and k

Arithmetic of Matrices

- The **Sum A + B** of two matrices **A**=[a_{jk}] and **B**=[b_{jk}] of the same size is obtained by adding the corresponding entries.

- **Scalar Multiplication** (Multiplication by a number)
  If **A**=[a_{jk}] and c is a number then c**.A** = [c.a_{jk}]

  - (**-**1)·**A** is written –**A** and is called the **negative** of **A**

  - **A**+(**-B**) is written as **A-B** and is called the **difference** of **A** and **B**

- An ***m* x *n* zero matrix** is an *m* x *n* matrix with all entries zero

Arithmetic of Matrices

- In Summary, for matrices of the same size we have:
  - $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - $(\mathbf{U} + \mathbf{V}) + \mathbf{W} = \mathbf{U} + (\mathbf{V} + \mathbf{W}) = \mathbf{U} + \mathbf{V} + \mathbf{W}$
  - $\mathbf{A} + \mathbf{0} = \mathbf{A}$
  - $\mathbf{A} + (\mathbf{-A}) = \mathbf{0} = \mathbf{A} - \mathbf{A}$
  - $c\cdot(\mathbf{A} + \mathbf{B}) = c\cdot\mathbf{A} + c\cdot\mathbf{B}$
  - $(c + k)\cdot\mathbf{A} = c\cdot\mathbf{A} + k\cdot\mathbf{A}$
  - $c\cdot(k\cdot\mathbf{A}) = (c\cdot k)\cdot\mathbf{A} = c\cdot k\cdot\mathbf{A}$
  - $1\cdot\mathbf{A} = \mathbf{A}$
  - $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
  - $(c\cdot\mathbf{A})^T = c\cdot\mathbf{A}^T$

Matrix Multiplication

- The product **C** = **A**·**B** (in this order) of an *m* x *n* matrix **A**=[a_{jk}] and an *r* x *p* matrix **B**=[b_{jk}] is defined if and only if *r*=*n* and the matrix **C** is *m* x *p* with entries c_{jk} are given by:

$$c_{jk} = \sum_{h=1}^{n} a_{jh}b_{hk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \dots + a_{jn}b_{nk}$$

- **A·B ≠ B·A** in general
- **A·B** = **0** does **not** necessarily imply **A=0** or **B=0** or **B·A= 0**

e.g.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Matrix Multiplication

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
. & . & \cdots & \cdots \\
a_{j1} & a_{j2} & \cdots & a_{jn} \\
\vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}_{m\times n}
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1p} \\
b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nk} & \cdots & b_{np}
\end{bmatrix}_{np}
=
\begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1k} & \cdots & c_{1p} \\
c_{21} & c_{22} & \cdots & c_{2k} & \cdots & c_{2p} \\
. & . & \cdots & . & \cdots & . \\
c_{j1} & c_{j2} & \cdots & c_{jk} & \cdots & c_{jp} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mk} & \cdots & c_{mp}
\end{bmatrix}_{m\times p}
$$

j-th row ● *m*-th column = *jk*-th element

Order of Multiplication

- The order of matrix multiplication is important. To emphasize this we say that in $\mathbf{A}\cdot\mathbf{B}$, the matrix **B** is the **pre-multiplied** or **multiplied from the left** by **A** and **A** is **post-multiplied** or **multiplied from the right** by **B**

- In Summary we have:
  - $(k\cdot\mathbf{A})\cdot\mathbf{B} = k\cdot(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot(k\cdot\mathbf{B})$ written $k\cdot\mathbf{A}\cdot\mathbf{B}$ or $\mathbf{A}\cdot k\cdot\mathbf{B}$
  - $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$ written $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$
  - $(\mathbf{A} + \mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C} + \mathbf{B}\cdot\mathbf{C}$ keep the order
  - $\mathbf{C}\cdot(\mathbf{A} + \mathbf{B}) = \mathbf{C}\cdot\mathbf{A} + \mathbf{C}\cdot\mathbf{B}$

provided **A**,**B**,**C** are such that the expressions on the left are defined and k is a scalar.

Special Matrices

- **Triangular matrices –**
  - a square matrix whose entries above the main diagonal are all zero is called a **lower triangular matrix**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

  - and a square matrix whose entries below the main diagonal are all zero is called a **upper triangular matrix**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

e.g.

lower triangular

$$\mathbf{T_1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 5 & 0 & 2 \end{bmatrix}$$

upper triangular

$$\mathbf{T_2} = \begin{bmatrix} 1 & 6 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

- An entry on the main diagonal may be zero or not

Special Matrices

- **Diagonal matrices –**
  - If **A** = [a_{jk}] is a square matrix with $a_{jk}=0 \ \forall\ j \neq k$ then **A** is called a **diagonal matrix**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

  - A diagonal matrix whose entries on the main diagonal are all equal is called a **scalar matrix**

  - If S is an *n* x *n* scalar matrix

$$\mathbf{S} = \begin{bmatrix} c & 0 & \cdots\cdots & 0 \\ 0 & c & \cdots\cdots & 0 \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ 0 & 0 & \cdots\cdots & c \end{bmatrix}$$

$$\mathbf{A}\cdot\mathbf{S} = \mathbf{S}\cdot\mathbf{A} = c\cdot\mathbf{A}$$

for any *n* x *n* matrix **A**

  - A scalar matrix whose entries on the main diagonal are all 1 is called a **unit matrix** and is denoted by $\mathbf{I}_n$ or **I** clearly

$$\mathbf{A}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{A} = \mathbf{A} \qquad \text{if } \mathbf{A}, \mathbf{I} \text{ are both } n \times n.$$

Properties of the Product

- **Transpose of a Product**:

$$(\mathbf{A}\cdot\mathbf{B})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\cdot\mathbf{A}^{\mathbf{T}}$$

- **Inner Product of Vectors**:

if $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are column vectors with *n* components then the **inner product** or **dot product** of $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$, denoted $\bar{\mathbf{a}}\cdot\bar{\mathbf{b}}$ is defined by:

$$\bar{\mathbf{a}}\cdot\bar{\mathbf{b}} = \bar{\mathbf{a}}^{\mathrm{T}}\bar{\mathbf{b}} = [a_1.....a_n]\cdot\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{h=1}^{n} a_h b_h = a_1 b_1 + \ldots + a_n b_n$$

Product in Terms of Vectors

- An *m* x *n* matrix **A**=[a_{jk}] can be written as a column matrix in terms of its row vectors

$$\mathbf{A} = \begin{bmatrix} \bar{\mathbf{a}}_1 \\ \bar{\mathbf{a}}_2 \\ \vdots \\ \bar{\mathbf{a}}_m \end{bmatrix} \quad \text{where} \quad \begin{aligned} \bar{\mathbf{a}}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \bar{\mathbf{a}}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ &\vdots \\ \bar{\mathbf{a}}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

- Similarly an *n* x *p* matrix **B**=[b_{jk}] can be written as a row matrix in terms of its column vectors

$$\mathbf{B} = \begin{bmatrix} \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \cdots & \bar{\mathbf{b}}_p \end{bmatrix} \text{ where } \bar{\mathbf{b}}_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}, \bar{\mathbf{b}}_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix}, \cdots, \bar{\mathbf{b}}_p = \begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{np} \end{bmatrix}$$

Product in Terms of Vectors

- Then the product **C=A·B** can be written

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} \overline{\mathbf{a}}_1 \cdot \overline{\mathbf{b}_1} & \overline{\mathbf{a}}_1 \cdot \overline{\mathbf{b}_2} & \cdots & \overline{\mathbf{a}}_1 \cdot \overline{\mathbf{b}_p} \\ \overline{\mathbf{a}}_2 \cdot \overline{\mathbf{b}_1} & \overline{\mathbf{a}}_2 \cdot \overline{\mathbf{b}_2} & \cdots & \overline{\mathbf{a}}_2 \cdot \overline{\mathbf{b}_p} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\mathbf{a}}_m \cdot \overline{\mathbf{b}_1} & \overline{\mathbf{a}}_m \cdot \overline{\mathbf{b}_2} & \cdots & \overline{\mathbf{a}}_m \cdot \overline{\mathbf{b}_p} \end{bmatrix}$$

Also we have:

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} \mathbf{A} \cdot \overline{\mathbf{b}_1} & \mathbf{A} \cdot \overline{\mathbf{b}_2} & \cdots & \mathbf{A} \cdot \overline{\mathbf{b}_p} \end{bmatrix}$$