

### ERG 2012B Advanced Engineering Mathematics II

#### Part II: Linear Algebra

#### Lecture #11 Matrices and Linear Equations

# Linear Algebra

• Matrix: a rectangular array of numbers (or functions), called entries or elements of the matrix, enclosed in brackets.

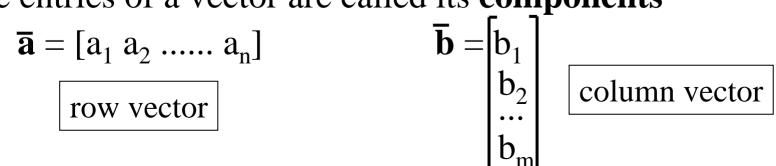
e.g. 
$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$
 rows: horizontal lines  
the matrix  $\mathbf{M}$  has 3 columns and 2 rows

- An *m* x *n* matrix *m* rows and *n* columns
- Double Subscript notation for matrix entries
  - •1<sup>st</sup> subscript denotes the row
  - •2<sup>nd</sup> subscript denotes column

$$\mathbf{A_{m'n}} = [\mathbf{a_{jk}}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{\mathbf{m'n}}$$

#### Matrices and Vectors

- If an *m* x *n* matrix has *m*=*n*, it is called an *n* x *n* square matrix
  - The entries  $a_{11}$ ,  $a_{22}$ ,...., $a_{nn}$  are called the **main** or **principal diagonal** of the matrix A
- A **submatrix** of and *m* x *n* matrix **A** is a matrix obtained by omitting some rows or columns (or both) from **A**
- For convenience, this includes **A** itself (i.e.  $\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
- Vectors: column/row vectors are a single column/row matrix The entries of a vector are called its **components**



# Transposition of a Matrix

- The **transpose**  $A^T$  of an  $m \ge n$  matrix  $A=[a_{jk}]$  is the  $n \ge m$ matrix that has the first rows of A as its first column, the second row of A as its second column and so on.
- The transpose of A is:

$$\mathbf{A} = [\mathbf{a}_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{A}^{\mathrm{T}} = [\mathbf{a}_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & \dots & a_{nm} \end{bmatrix}$$

• Example  $\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix} \qquad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$ 

•The transpose of a column vector is a row vector and vice versa

## Symmetric Matrices

 A Symmetric Matrix is a square matrix whose transpose is the same as the original matrix – i.e. A<sup>T</sup> = A

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -3 \\ -2 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}^{T} = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

- A Skew-symmetric Matrix is a square matrix whose transpose is the same as the negative of the original matrix – i.e. A<sup>T</sup> = -A
- Two Matrices A = [a<sub>jk</sub>] and B = [b<sub>jk</sub>] are equal, written A=B, if and only if they have the same size and the corresponding entries are all equal; i.e. a<sub>jk</sub> = b<sub>jk</sub> for every j and k

## Arithmetic of Matrices

- The **Sum A** + **B** of two matrices **A**=[a<sub>jk</sub>] and **B**=[b<sub>jk</sub>] of the same size is obtained by adding the corresponding entries.
- Scalar Multiplication (Multiplication by a number) If  $\mathbf{A} = [a_{jk}]$  and c is a number then  $c \cdot \mathbf{A} = [c \cdot a_{jk}]$ 
  - (-1)•A is written –A and is called the **negative** of A
  - A+(-B) is written as A-B and is called the **difference** of A and B
- An *m* x *n* zero matrix is an *m* x *n* matrix with all entries zero

## Arithmetic of Matrices

- In Summary, for matrices of the same size we have:
  - $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - (U + V) + W = U + (V + W) = U + V + W
  - $\mathbf{A} + \mathbf{0} = \mathbf{A}$
  - $\mathbf{A} + (-\mathbf{A}) = \mathbf{0} = \mathbf{A} \mathbf{A}$
  - $\mathbf{c} \cdot (\mathbf{A} + \mathbf{B}) = \mathbf{c} \cdot \mathbf{A} + \mathbf{c} \cdot \mathbf{B}$
  - $(c + k) \cdot A = c \cdot A + k \cdot A$
  - $\mathbf{c} \cdot (\mathbf{k} \cdot \mathbf{A}) = (\mathbf{c} \cdot \mathbf{k}) \cdot \mathbf{A} = \mathbf{c} \cdot \mathbf{k} \cdot \mathbf{A}$
  - $1 \cdot \mathbf{A} = \mathbf{A}$
  - $(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$
  - $(\mathbf{c} \cdot \mathbf{A})^{\mathrm{T}} = \mathbf{c} \cdot \mathbf{A}^{\mathrm{T}}$

# Matrix Multiplication

The product C = A·B (in this order) of an m x n matrix A=[a<sub>jk</sub>] and an r x p matrix B=[b<sub>jk</sub>] is defined if and only if r=n and the matrix C is m x p with entries c<sub>ik</sub> are given by:

$$c_{jk} = \sum_{h=1}^{n} a_{jh} b_{hk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \dots + a_{jn} b_{nk}$$

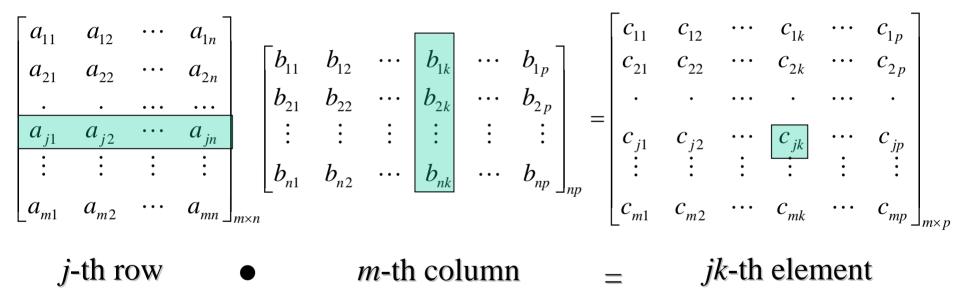
- $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$  in general
- $\mathbf{A} \cdot \mathbf{B} = \mathbf{0}$  does **not** necessarily imply  $\mathbf{A} = \mathbf{0}$  or  $\mathbf{B} = \mathbf{0}$  or  $\mathbf{B} \cdot \mathbf{A} = \mathbf{0}$

e.g. but  

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} =
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} =
\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$



#### Matrix Multiplication



# Order of Multiplication

- The order of matrix multiplication is important. To emphasize this we say that in A·B, the matrix B is the pre-multiplied or multiplied from the left by A and A is post-multiplied or multiplied from the right by B
- In Summary we have:
  - $(\mathbf{k} \cdot \mathbf{A}) \cdot \mathbf{B} = \mathbf{k} \cdot (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot (\mathbf{k} \cdot \mathbf{B})$
  - $\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$
  - $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$

- written k·A·B or A·k·B
- written A·B·C
- keep the order

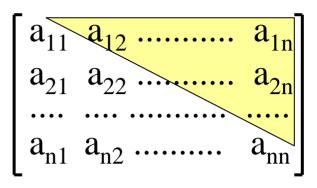
•  $\mathbf{C} \cdot (\mathbf{A} + \mathbf{B}) = \mathbf{C} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{B}$ 

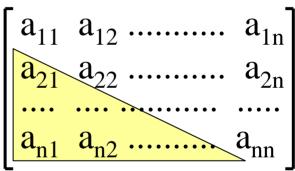
provided **A**,**B**,**C** are such that the expressions on the left are defined and k is a scalar.

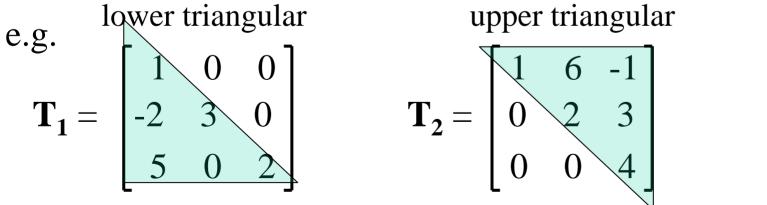


# Special Matrices

- Triangular matrices
  - a square matrix whose entries above the main diagonal are all zero is called a **lower triangular matrix**
  - and a square matrix whose entries below the main diagonal are all zero is called a **upper triangular matrix**





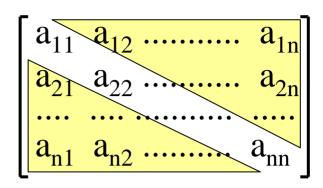


• An entry on the main diagonal may be zero or not

# Special Matrices



If A = [a<sub>jk</sub>] is a square matrix with
 a<sub>jk</sub>=0 ∀ j≠k then A is called a
 diagonal matrix



- A diagonal matrix whose entries on the main diagonal are all equal is called a **scalar matrix**
- If S is an  $n \ge n$  scalar matrix  $\mathbf{A} \cdot \mathbf{S} = \mathbf{S} \cdot \mathbf{A} = \mathbf{c} \cdot \mathbf{A}$ for any  $n \ge n$  matrix  $\mathbf{A}$

 $\mathbf{S} = \begin{bmatrix} c & 0 & \dots & 0 \\ 0 & c & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & c \end{bmatrix}$ 

• A scalar matrix whose entries on the main diagonal are all 1 is called a **unit matrix** and is denoted by  $I_n$  or I clearly

 $\mathbf{A} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{A} = \mathbf{A}$  if  $\mathbf{A}$ ,  $\mathbf{I}$  are both  $n \ge n$ .



#### Properties of the Product

• Transpose of a Product:

 $(\mathbf{A} \boldsymbol{\cdot} \mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} \boldsymbol{\cdot} \mathbf{A}^{\mathrm{T}}$ 

• Inner Product of Vectors:

if  $\overline{\mathbf{a}}$  and  $\overline{\mathbf{b}}$  are column vectors with *n* components then the **inner product** or **dot product** of  $\overline{\mathbf{a}}$  and  $\overline{\mathbf{b}}$ , denoted  $\overline{\mathbf{a}} \cdot \overline{\mathbf{b}}$  is defined by:

$$\mathbf{\bar{a}} \cdot \mathbf{\bar{b}} = \mathbf{\bar{a}}^{\mathbf{T}} \mathbf{\bar{b}} = [a_1 \dots a_n] \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{h=1}^n a_h b_h = a_1 b_1 + \dots + a_n b_n$$

#### Product in Terms of Vectors

• An *m* x *n* matrix  $\mathbf{A} = [a_{jk}]$  can be written as a column matrix in terms of its row vectors

$$\mathbf{A} = \begin{bmatrix} \overline{\mathbf{a}}_{1} \\ \overline{\mathbf{a}}_{2} \\ \vdots \\ \overline{\mathbf{a}}_{m} \end{bmatrix} \text{ where } \begin{bmatrix} \overline{\mathbf{a}}_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$
$$\text{ where } \begin{bmatrix} \overline{\mathbf{a}}_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$
$$\text{ is } \begin{bmatrix} \overline{\mathbf{a}}_{m} \\ \overline{\mathbf{a}}_{m} \end{bmatrix} = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

• Similarly an *n* x *p* matrix  $\mathbf{B} = [\mathbf{b}_{jk}]$  can be written as a row matrix in terms of its column vectors

$$\mathbf{B} = \begin{bmatrix} \overline{\mathbf{b}}_1 & \overline{\mathbf{b}}_2 & \cdots & \overline{\mathbf{b}}_p \end{bmatrix} \text{ where } \overline{\mathbf{b}}_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}, \overline{\mathbf{b}}_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix}, \cdots, \overline{\mathbf{b}}_p = \begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{np} \end{bmatrix}$$



#### Product in Terms of Vectors

• Then the product  $C=A\cdot B$  can be written

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} \overline{\mathbf{a}}_1 \cdot \overline{\mathbf{b}}_1 & \overline{\mathbf{a}}_1 \cdot \overline{\mathbf{b}}_2 & \cdots & \overline{\mathbf{a}}_1 \cdot \overline{\mathbf{b}}_p \\ \overline{\mathbf{a}}_2 \cdot \overline{\mathbf{b}}_1 & \overline{\mathbf{a}}_2 \cdot \overline{\mathbf{b}}_2 & \cdots & \overline{\mathbf{a}}_2 \cdot \overline{\mathbf{b}}_p \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\mathbf{a}}_m \cdot \overline{\mathbf{b}}_1 & \overline{\mathbf{a}}_m \cdot \overline{\mathbf{b}}_2 & \cdots & \overline{\mathbf{a}}_m \cdot \overline{\mathbf{b}}_p \end{bmatrix}$$

Also we have:  $\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} \mathbf{A} \cdot \overline{\mathbf{b}}_1 & \mathbf{A} \cdot \overline{\mathbf{b}}_2 & \cdots & \mathbf{A} \cdot \overline{\mathbf{b}}_p \end{bmatrix}$