ERG 2012B
Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #10

Singularities, Zeros and Residue Integration

Singularities, Zeros and Infinity

- We say that a function f(z) is **singular** or has a **singularity** at a point $z=z_0$ if f(z) is not analytic (maybe even undefined) at $z=z_0$ but every neighbourhood of $z=z_0$ contains points at which f(z) is analytic

- We call $z=z_0$ an **isolated singularity** of f(z) if $z=z_0$ has a neighbourhood without further singularities of f(z)

- Isolated singularities of f(z) at $z=z_0$ can be classified by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=0}^{\infty} b_n(z-z_0)^{-n}$$

valid in the *immediate neighbourhood* of the singular point $z=z_0$ except at z_0 itself (in the region $0 < |z-z_0| < R$)

Singularities, Zeros and Infinity

- The sum of the first series is analytic at $z=z_0$. The second series containing the negative powers, is called the **principal part**.

- If it has only a finite number of terms, it is of the form:

$$b_1/(z-z_0) + b_2/(z-z_0)^2 + \ldots\ldots.. + b_m/(z-z_0)^m$$

  then the singularity of f(z) at $z=z_0$ is called a **pole** and **m** is called its **order**

- Poles of the first order are also known as **simple poles**

- If the principal part has infinitely many terms f(z) is said to have an **isolated essential singularity** at $z=z_0$.

Examples

- The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at z=0 and a pole of fifth order at z=2

- The function

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!/z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots\ldots.$$

has an isolated essential singularity at z=0

- The function

$$\sin(1/z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, z^{2n+1}} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} \ldots\ldots.$$

also has an isolated essential singularity at z=0

- $f(z) = z^{-5}\sin z = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040}z^2 - \cdots$ has a 4^{th} order pole at z=0

Theorems

If $f(z)$ is analytic and has a pole at $z=z_0$, then $|f(z)|\to\infty$ as $z\to z_0$ in any manner.

Picard's Theorem If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value in an arbitrarily small neighbourhood of z_0

Removable Singularities A function is said to have a removable singularity at $z=z_0$ if $f(z)$ is not analytic at $z=z_0$, but can be made analytic there by assigning a suitable value of $f(z_0)$

e.g. $f(z) = (\sin z)/z$ becomes analytic at $z=0$ if we define $f(0)=1$

Zeros of Analytic Functions

- An analytic function f(z) in some domain D is said to have a **zero** at $z=z_0$ in D if $f(z_0)=0$.
- This zero is said to be of **order n** if not only f but also the derivatives $f^{/}, f^{//},.....f^{(n-1)}$ are all zero at $z=z_0$ but $f^{(n)}(z_0)\neq 0$
- A zero of first order is called a **simple zero**
- The Taylor series of f(z) **at a zero of nth order** is given by $f(z) = (z-z_0)^n[a_n+a_{n+1}(z-z_0)+a_{n+2}(z-z_0)^2+....] \quad a_n\neq 0$

Theorem: The zeros of an analytic function f(z) are isolated i.e. each of them has a neighbourhood that contains no further zeros of f(z)

Theorem: Let f(z) be analytic at $z=z_0$ and have a zero of the n^{th} order at $z=z_0$. Then 1/f(z) has a pole of the n^{th} order at $z=z_0$. The same holds for h(z)/f(z) if h(z) is analytic at $z=z_0$ and $h(z_0)\neq 0$

Residues

- If f(z) has a Laurent series near $z=z_0$ except at $z=z_0$ itself

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n/(z-z_0)^n$$

- The coefficient b_1 is called the **residue** of f(z) at $z=z_0$ and is denoted by

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

- Remember that:

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)dz$$

- As b_1 can be found without using the integral formula, we have a useful method of evaluating integrals – **the residue integration method**

Example 1

Integrate $f(z) = z^{-4}\sin z$ counterclockwise around the unit circle C

Solution: The Laurent series of f(z) is:

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \ldots\ldots\ldots \qquad |z| > 0$$

$$\operatorname*{Res}_{z=0} f(z) = b_1 = -\frac{1}{3!} = -\frac{1}{6}$$

Therefore

$$\oint_C \frac{\sin z}{z^4}\,dz = 2\pi i\, b_1 = -\frac{\pi i}{3}$$

Example 2

Integrate $f(z) = 1/(z^3 - z^4)$ ccw around the circle C: $|z| = \frac{1}{2}$

Solution:

$$f(z) = \frac{1}{z^3}\,\frac{1}{(1-z)}$$

has a simple pole at z=1 and a third order pole at z=0

The pole at z=1 is outside C and is irrelevant.

We need to find $\operatorname*{Res}_{z=0} f(z)$

$$\text{Since } f(z) = \frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)} = \frac{1}{z^3}(1 + z + z^2 + ...) \qquad 0<|z|<1$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + ...$$

Therefore

$$\oint \frac{dz}{z^3 - z^4} = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i$$

Residues at Simple Poles

If f(z) has a simple pole at $z=z_0$, then

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0)f(z)$$

Proof:

Since $f(z) = \dfrac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\ldots$

Multiply both sides by $(z-z_0)$ $(b_1 \neq 0,\ 0<|z-z_0|<R)$

$$(z-z_0)f(z) = b_1 + (z-z_0)[a_0 + a_1(z-z_0) + \ldots]$$

now let $z \to z_0$

Example: $$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \lim_{z\to i}(z-i)\frac{9z+i}{z(z+i)(z-i)}$$

$$= [\frac{9z+i}{z(z+i)}]_{z=i} = \frac{10i}{-2} = -5i$$

Quotients

If $f(z) = p(z)/q(z)$ with analytic $p(z)$ and $q(z)$ and $p(z_0) \neq 0$ but $q(z)$ has a simple zero at $z=z_0$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} p(z)/q(z) = p(z_0)/q'(z_0)$$

Proof: By assumption

$$q(z) = (z-z_0)q'(z_0) + \frac{(z-z_0)^2}{2!}q''(z_0) + \ldots\ldots.$$

and $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0)\, p(z)/q(z)$

$$= \lim_{z \to z_0} \frac{(z-z_0)p(z)}{(z-z_0)[q'(z_0) + (z-z_0)q''(z_0)/2 + \ldots.]} = \frac{p(z_0)}{q'(z_0)}$$

Example: $$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \left[\frac{9z+i}{3z^2+1}\right]_{z=i} = \frac{10i}{-2} = -5i$$

Residue at a Pole of Any Order

Let f(z) be an analytic function that has a pole of order m>1 at $z=z_0$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \right\}$$

In particular, for a 2nd order pole (m=2)

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \{[(z-z_0)^2 f(z)]'\}$$

Example: $$\operatorname*{Res}_{z=1} \frac{50z}{(z+4)(z-1)^2} = \lim_{z\to 1} \frac{d}{dz}[(z-1)^2 f(z)]$$

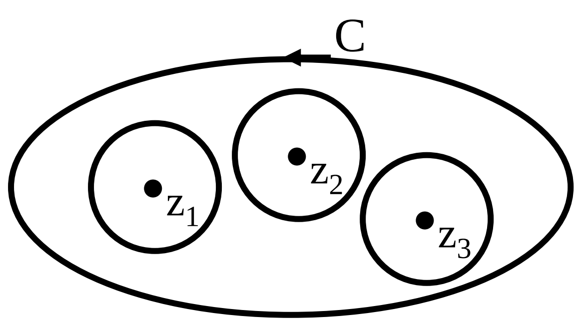
$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{50z}{z+4}\right] = 8$$

Residue Theorem

Let f(z) be a function that is analytic inside and on a simple closed path C, except for finitely many singular point z_1, z_2, ...z_k inside C. Then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1} \operatorname*{Res}_{z=z_j} f(z)$$

the integral being taken ccw around C



Example 1

Evaluate $I = \oint_C \frac{4-3z}{z^2-z}\,dz$, where C is any simple closed path ccw such that a) 0 and 1 are inside C, b) 0 is inside, 1 outside,

c) 1 is inside; 0 outside d) 0 and 1 are outside.

Solution: The integrand has simple poles at 0 and 1

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4$$

$$\operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1$$

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z-z_0)f(z)$$

a) $I = 2\pi i\,[-4 + 1] = -6\,\pi i$

b) $I = 2\pi i\,[-4] = -8\,\pi i$

c) $I = 2\pi i\,[1] = 2\,\pi i$

d) $I = 0$

Evaluation of Real Integrals

Consider integrals of the type:

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

Let $z = e^{i\theta}$ then we can use

$\cos\theta = ½(e^{i\theta} + e^{-i\theta}) = ½(z+1/z)$

$\sin\theta = ½(e^{i\theta} - e^{-i\theta}) = ½(z-1/z)$

and substitute $f(z)$ for $F(\cos\theta, \sin\theta)$.

As θ ranges from 0 to 2π, z ranges once ccw around circle $|z|=1$

$dz/d\theta = ie^{i\theta} = iz$ or $d\theta = dz/iz$

hence $I = \oint_C f(z)/iz\, dz$ - C is taken ccw around the unit circle.

Example

Evaluate

$$I = \int_0^{2\pi} 1/(\sqrt{2}-\cos\theta)\, d\theta$$

Solution: Let $z=e^{i\theta}$ then

$$I = \oint_C \frac{1}{iz(\sqrt{2}-½(z+1/z))}\, dz = \oint_C \frac{2}{i(z^2-2\sqrt{2}z+1)}\, dz$$

$$= -\frac{2}{i}\oint_C \frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}\, dz = -\frac{2}{i}\oint_C \frac{1}{(z-z_1)(z-z_2)}\, dz$$

where $z_1=\sqrt{2}+1$, $z_2=\sqrt{2}-1$

since $|z_1| > 1$, $|z_2| < 1$

$$I = -\frac{2}{i}\, 2\pi i \operatorname*{Res}_{z=z_2}\left[\frac{1}{(z-z_1)(z-z_2)}\right] = -\left[\frac{1}{2z-(z_1+z_2)}\right]_{z=z^2} = -4\pi(-½) = 2\pi$$