ERG 2012B
Advanced Engineering Mathematics II

Part I: Complex Variables

Lecture #1: Complex Numbers

Complex Numbers

Historically, i is introduced as the root of the equation $z^2 = -1$

$$i = \sqrt{-1} \text{ and } i^2 = -1$$

Definition

A complex number, z, is an ordered pair of real numbers, x and y, written

$$z = (x,y)$$

x is called the ***real part*** of z - Re(z)

y is the ***imaginary part*** of z - Im(z)

Complex Numbers

Definition

by definition two complex numbers are **equal** if and only if their real parts are equal **and** their imaginary parts are equal

Definition

(0,1) is called the imaginary unit and is denoted by

$$i = (0,1)$$

(Often denoted as "j" in electrical engineering to avoid confusion with symbol for current)

Complex Numbers

Definition

A complex number $z = (x,y)$ is commonly written as $z = x + i\, y$

if $x = 0$ (i.e. $z = i\, y$) then z is called *pure imaginary*

Complex Conjugate

Definition

The ***complex conjugate*** of $z = x + i\, y$ is defined as

$$\overline{z} = x - i\, y$$

Note:

$z * \overline{z} = (x+i\, y)(x-i\, y) = x^2+y^2$ (like: $(x+y)(x-y) = x^2-y^2$)

Addition & Multiplication

Addition & Multiplication of Complex Numbers

Let $z_1 = x_1 + i\, y_1$ and $z_2 = x_2 + i\, y_2$

by definition:

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)$$

subtraction and **division** are defined as the inverse operations of addition and multiplication respectively

Difference

Definition

So that the difference $z = z_1 - z_2$ is defined such that

$$z + z_2 = z_1$$

Hence

$$z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$$

Quotient

Definition

And the quotient $z = z_1/z_2$ $(z_2 \neq 0)$ is defined such that

$$z\, z_2 = z_1$$

It can be shown that:

$$z_1/z_2 = (x_1x_2+y_1y_2)/(x_2^2+y_2^2) + i\,(x_2y_1-x_1y_2)/(x_2^2+y_2^2)$$

Quotient

In practice, the quotient is often found by multiplying the numerator and denominator of z_1/z_2 by the complex conjugate of z_2

i.e.

$z = (x_1+iy_1)/(x_2+iy_2) = (x_1+iy_1)(x_2-iy_2)/(x_2+iy_2)(x_2-iy_2)$

or

$z=(x_1x_2+y_1y_2)/(x_2^2+y_2^2)+ i\ (x_2y_1-x_1y_2)/(x_2^2+y_2^2)$

Properties

The Properties of Addition and multiplication for complex numbers are the same as for real numbers:

Commutative Law:

$$z_1+z_2 = z_2+z_1$$

$$z_1 z_2 = z_2 z_1$$

Associative Law:

$$(z_1+z_2)+z_3 = z_1+(z_2+z_3)$$

$$(z_1 z_2)z_3 = z_1(z_2 z_3)$$

Properties

The Properties of Addition and multiplication for complex numbers are the same as for real numbers:

Distributive Law:

$$z_1(z_2+z_3) = z_1z_2+z_1z_3$$

and

$0+z = z+0 = z$	(identity)
$z+(-z) = (-z)+z = 0$	(inverse)
$z.1 = 1.z = z$	(identity)

Properties - Conjugates

From the definition of the complex conjugate it is clear that:

$$\mathrm{Re}(z) = (z+\overline{z})/2$$

$$\mathrm{Im}(z) = (z-\overline{z})/2i$$

and

$$\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$$

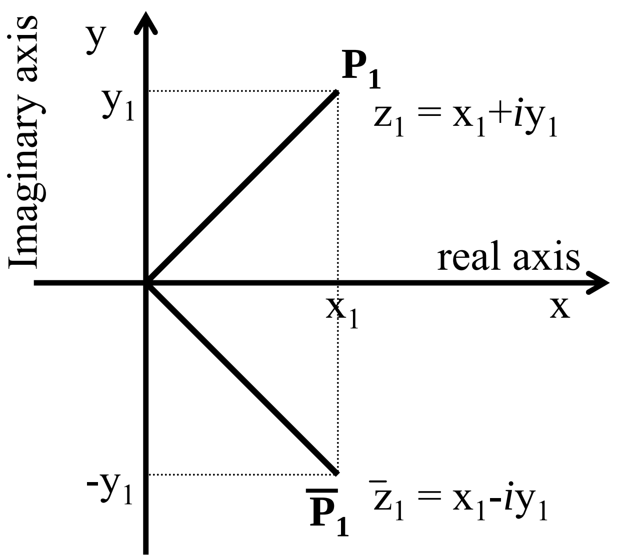
$$\overline{z_1-z_2} = \overline{z_1} - \overline{z_2}$$

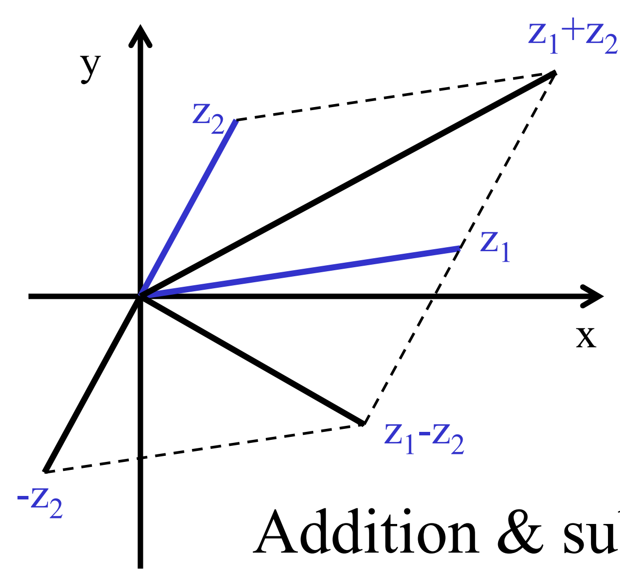
$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$$

$$\overline{(z_1/z_2)} = \overline{z_1} / \overline{z_2}$$

Graphical Representation

Complex numbers can be represented geometrically as a point in a plane – ***the complex plane*** – with the horizontal x-axis called the ***real axis*** and the vertical y-axis called the ***imaginary axis***





Addition & subtraction

Examples

Let $z_1 = 18 + 3i$ and $z_2 = -7 + 2i$

then

$Re(z_1) = 18$, $Re(z_2) = -7$

$Im(z_1) = 3$, $Im(z_2) = 2$

$z_1 + z_2 = 11 + 5i$, $z1 - z2 = 25 + i$

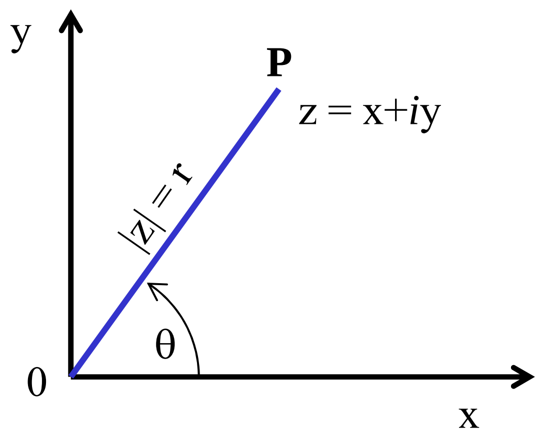
$z_1 z_2 = (18+3i)(-7+2i) = -132 + 15i$

$z_1/z_2 = (18+3i)(-7-2i)/(7^2+2^2) = (-120-57i)/53$

$\overline{z_1}\,\overline{z_2} = (18-3i)(-7-2i) = -132-15i = \overline{z_1 z_2}$

$\overline{z_1}/\overline{z_2} = (18-3i)(-7+2i)/(7^2+2^2) = (-120+57i)/53 = \overline{z_1/z_2}$

Polar Form of Complex Numbers



The **polar form** of $z=x+iy$ is

$$z = r(\cos\theta + i \sin\theta)$$

r is called the **absolute value** or **modulus** of z and is denoted by **|z|**

$$|z| = r = \sqrt{(x^2+y^2)} = \sqrt{(z\bar{z})}$$

θ is called the **argument** of z and is denoted by **arg z**

$$\theta = \arg z = \arctan (y/x)$$

- θ is normally measured in radians and positive in counterclockwise sense
- for $z = 0$, angle θ is undefined
- for a given z ($\neq 0$) determined up to integer multiples of 2π

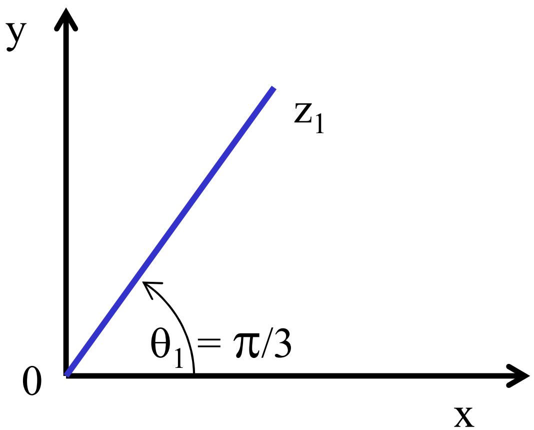
Polar Form of Complex Numbers

Definition

The value of θ that lies in the interval $-\pi < \theta \leq \pi$ is called the **principal value** of the argument of z and is denoted by **Arg z** which satisfies

$$-\pi < \text{Arg } z \leq \pi$$

Example 1



if $z_1 = 1 + \sqrt{3}\, i$

then $r_1 = \sqrt{(1+3)} = 2$

$\theta_1 = \tan^{-1}(\sqrt{3}/1) = \pi/3$

so $z_1 = 2(\cos(\pi/3) + i \sin(\pi/3))$

$|z_1| = 2$

$\arg z_1 = \pi/3 \pm 2n\pi$ $(n = 0,1,2.....)$

$\mathrm{Arg}\, z_1 = \pi/3$ - the principal value,

other values are $-5\pi/3$, $7\pi/3$, etc...

Example 2

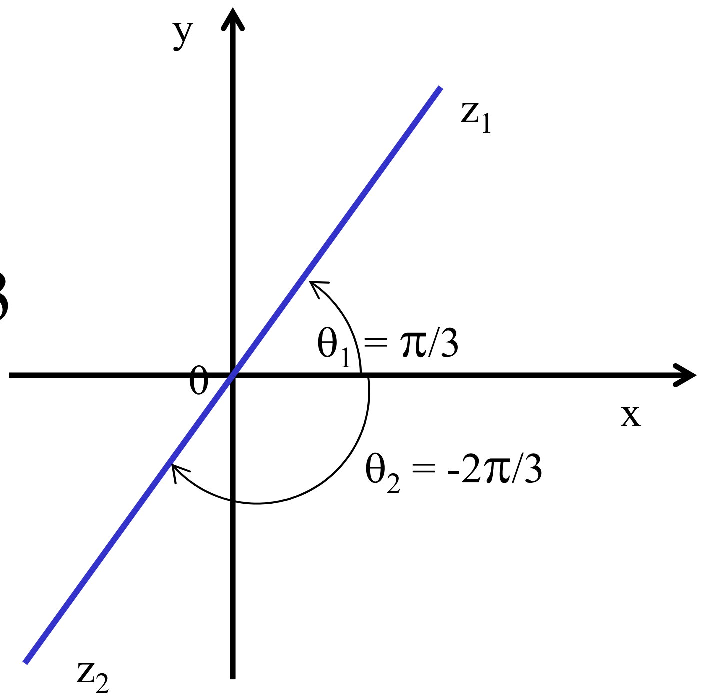
if $z_2 = -z_1 = -1 - \sqrt{3}\, i$

then $|z_2| = r_2 = \sqrt{(1+3)} = 2$

$\theta_2 = \tan^{-1}(-\sqrt{3}/-1) = -2\pi/3$

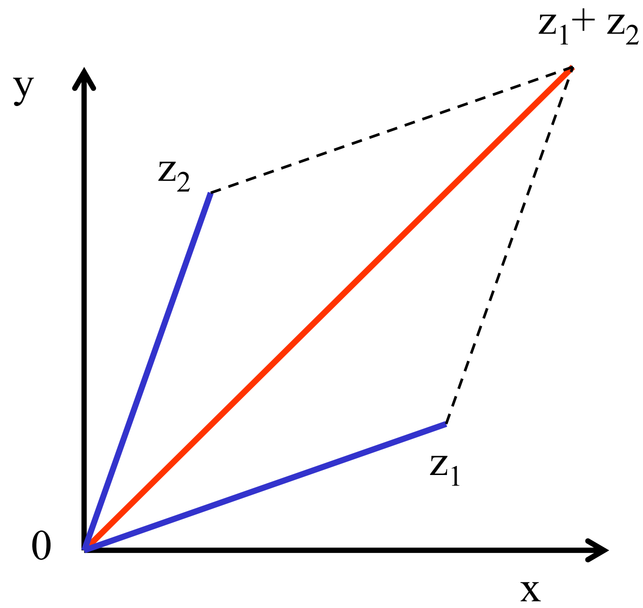
so

$\text{Arg } z_2 = -2\pi/3$



Note: $\tan(\theta_2) = \tan(\theta_1) = \sqrt{3}$, but z_2 lies in the 3rd quadrant so $\theta_2 = -2\pi/3$

Triangular Inequality



For any complex numbers

$$|z_1+z_2| \leq |z_1| + |z_2|$$

In general

$$|z_1+z_2+ \ldots\ldots +z_n| \leq |z_1|+|z_2|+ \ldots\ldots +|z_n|$$

Multiplication in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplication

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

$$= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$|z_1 z_2| = |z_1|\,|z_2|$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ (up to multiples of 2π)

Division in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\ \sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\ \sin\theta_2)$,

Division

$z = z_1/z_2$ then $z\ z_2 = z_1$

therefore $|z\ z_2| = |z|\ |z_2| = |z_1|$

$\arg(z\ z_2) = \arg(z) + \arg(z_2) = \arg(z_1)$

Hence $|z_1/z_2| = |z| = |z_1|/|z_2|$

$\arg(z_1/z_2) = \arg(z) = \arg(z_1)-\arg(z_2)$

or $z1/z2 = (r_1/r_2)[\cos(\theta_1-\theta_2)+i\ \sin(\theta_1-\theta_2)]$

Example

Let $z_1 = -2 + 2i$

$z_2 = 3i$

then $z_1 z_2 = -6 - 6i$

$z_1/z_2 = 2/3 + (2/3)i$

as:

$|z_1| = 2\sqrt{2}, \quad |z_2| = 3$

$|z_1|\,|z_2| = 6\sqrt{2} = |z_1\, z_2|$

$|z_1|/|z_2| = (2\sqrt{2})/3 = |z_1/z_2|$

$\text{Arg } z_1 = 3\pi/4, \ \text{Arg } z_2 = \pi/2$

$\text{Arg}(z_1\, z_2) = -3\pi/4 = \text{Arg } z_1 + \text{Arg } z_2 - 2\pi$

$\text{Arg}(z_1/z_2) = \pi/4 = \text{Arg } z_1 - \text{Arg } z_2$

Integer Powers of z

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$$

$$z^{-2} = r^{-2}[\cos(-2\theta) + i \sin(-2\theta)]$$

more generally

$$z^n = r^n[\cos(n\theta) + i \sin(n\theta)]$$

Formula of De Moivre:

for $|z| = r = 1$

$$z^n = (\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta)$$