Signal Processing Basics

A Preparatory Course for M.Sc Student

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This course prepares students planning to take

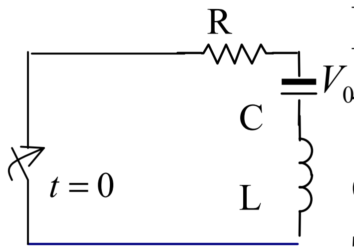
ELE 7080	Digital Processing of Speech Signals
ELE 7090	Image Processing and Video Technology
ELE 7100	Advanced Signal Processing for Communications

Contents: Linear System Analysis, Transforms, Digital Signal Processing, Analog Modulation

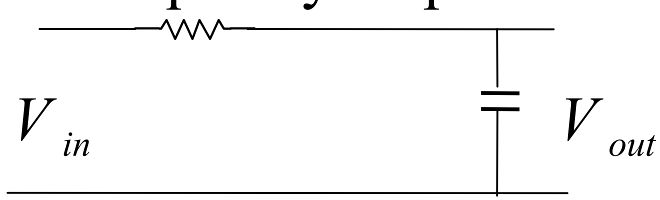
Duration: 2-5 pm on a Saturday afternoon

Do I need this course ?

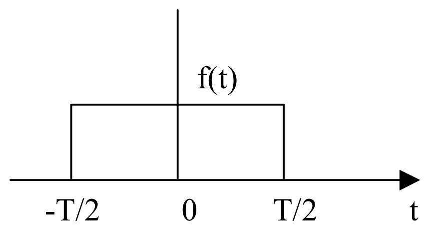
No, if you can answer most of the questions below, open book.

- 

Find the current in the circuit after closing of the switch at t=0. The initial voltage across the capacitor is V_o volts. Use the Laplace Transform method.
- Find the frequency response of the circuit



- Find the Fourier series (Trigonometric and Exponential) coefficients of
  - $f(t) = \cos\omega_0 t$
  - $f(t) = |\cos\omega_0 t|$

- Find the Fourier Transform of
  - 

  - $$f(t) = \begin{cases} \cos \omega_0 t & , -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & , otherwise \end{cases}$$

- A linear system has impulse response h(t), input x(t) and output y(t). Write the convolution equation relating these three quantities. Then take the Fourier Transform of this equation to prove the convolution theorem.
- Given two time functions a(t) and b(t), write the equations for their convolution and correlation.

- For the random process **x**(t), give two ways to calculate its power spectral density.
- The voltage across a 1Ω resistor is **x**(t), a zero mean white noise process. Find the autocorrelation and power spectrum of **x**(t). What physical quantity does the autocorrelation at zero shift represent?
- Let x(t) be a time function bandlimited to B Hz. What should be the minimum sampling frequency for x(t) to ensure no aliasing?
- Let

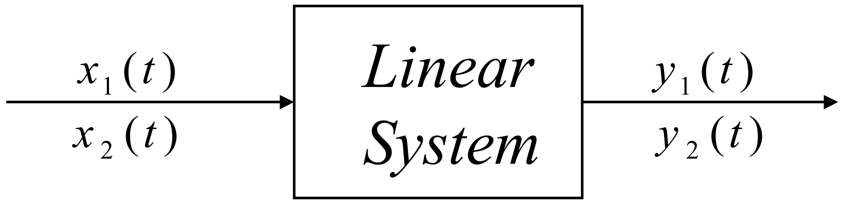
$$x(t) = \frac{\sin \omega_0 t}{\omega_0 t}$$

  - What is x(0)?
  - Is this a bandlimited signal?
  - Give the equation for the samples of x(t), taken at the Nyquist rate.

- Let $f(t)$ be samples of a function $f(nT)$, taken at a sampling frequency of $\frac{1}{T} = f_s$. Show a block diagram, and then the equation, for producing $f(t)$ from $f(nT)$.
- Write the Discrete Fourier Transform for the sample $f(nT)$.
- Let m(t) be the modulating signal and $\cos\omega_c t$ be the carrier. Write the equation for AM and FM.
- Draw the block diagram of an AM radio.
- Draw the spectrum of an AM signal when $m(t) = \cos \omega_m t$ and the carrier frequency is ω_c.

I.1 Linear System Relationships

Linear system definition.

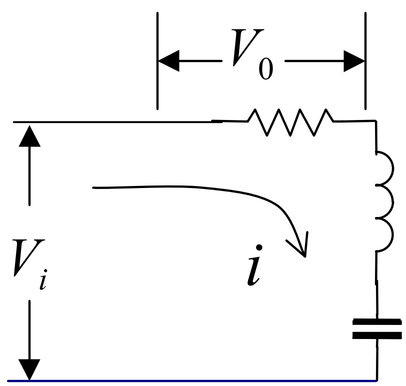


System is linear iff

$$\alpha\, x_1(t) + \beta\, x_2(t) = \alpha\, y_1(t) + \beta\, y_2(t) \quad , \ \alpha, \beta \ \text{ any constants}$$

A linear system satisfies Principles of Superposition

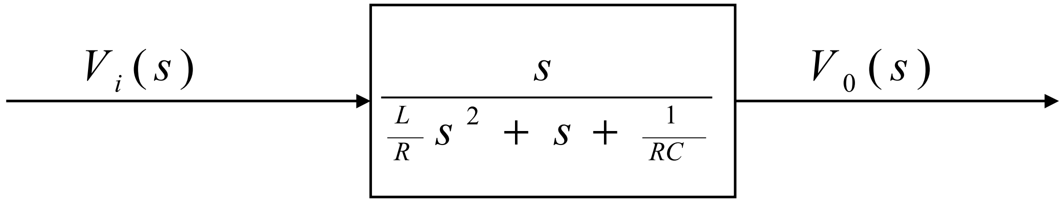
e.g.



$$V_i(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{c}\int i(t)dt$$

$$\frac{dV_i(t)}{dt} = \frac{dV_0(t)}{dt} + \frac{L}{R}\frac{d^2V_0(t)}{dt^2} + \frac{V_0(t)}{RC}$$

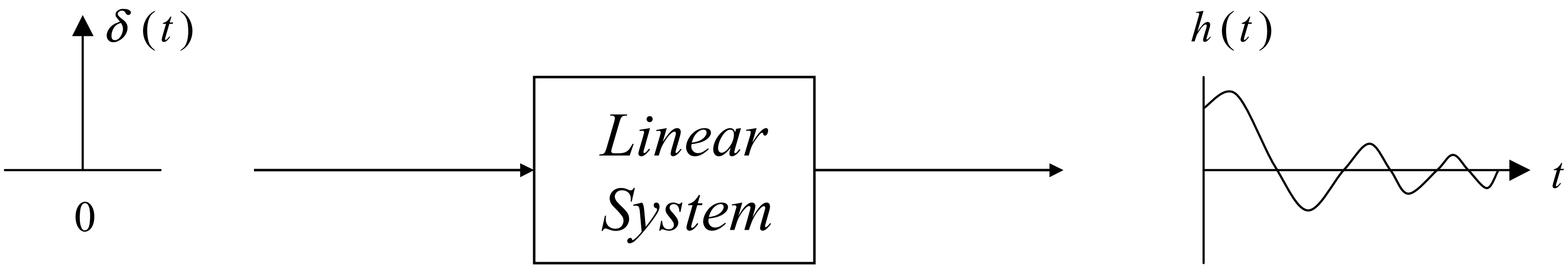
is linear



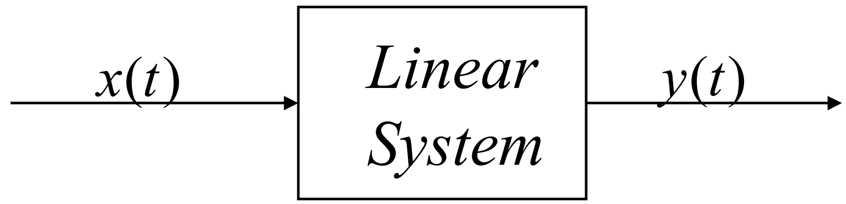
But

$$\frac{dV_i(t)}{dt} = 2\left(\frac{dV_0(t)}{dt}\right)^2 + 3\frac{dV_0(t)}{dt} \qquad \text{is nonlinear}$$

Impulse response and convolution

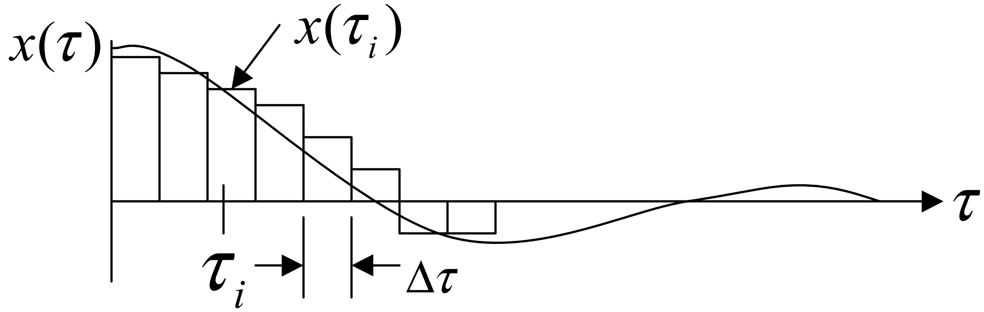


The output of a linear system when the input is a unit impulse is the unit impulse response h(t) of the system.



unit impulse response h(t)

Approximate $x(\tau)$ *by a series of pulses of height* $x(\tau_i)$ *and width* $\Delta\tau$



The output at time t , due to a pulse at τ_i *,is approximately*

$$x(\tau_i)\Delta\tau\, h(t-\tau_i)$$

By superposition, the total response at time t due to pulses at $\tau_i, i = 0,1,2,.......$ *is*

$$y(t) \cong \sum_{i=1}^{N} x(\tau_i)\, h(t-\tau_i)\Delta\tau$$

As $\Delta\tau \to d\tau, x(\tau_i) \to x(\tau)$, then

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Note that for causal systems, $h(t) \equiv 0 \quad for \quad t < 0$.

With a change of variable $\lambda = t - \tau$, $y(t)$ becomes

$$y(t) = \int_{-\infty}^{\infty} x(\tau - \lambda)\, h(\lambda) d\lambda = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau) d\tau = x(t) * h(t)$$

$* =$ *convolution. The output of a linear system with unit impulse response h(t), and input x(t), is the convolution of x(t) with h(t).*

Difference between convolution and correlation

Correlation between a(t), b(t) is

$$R_{ab}(\tau) = \frac{1}{2T} \int_{-T}^{T} a(t) b(t + \tau) dt$$

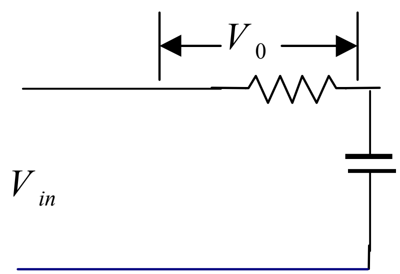
Convolution between a(t), b(t) is

$$a(t) * b(t) = \int_{-\infty}^{\infty} a(\lambda) b(t - \lambda) d\lambda$$

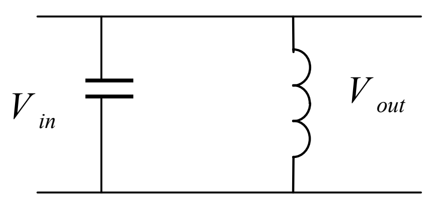
I.1 Review Question

- Verify that the system on P.7 is linear.
- Show that the system on P.8 is nonlnear.
- How do you produce an impulse ?
- What is the impulse response of the systems

a)



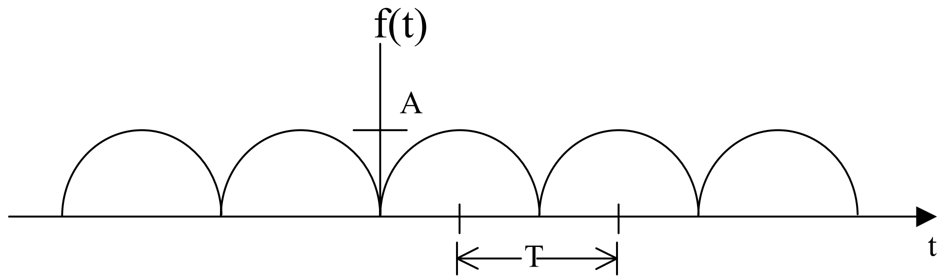
b)



I.2 Fourier Series

Fourier series expresses a periodic signal as a sum of sines and cosines, of different amplitudes and frequencies that are harmonics of the fundamental frequency ($= \frac{1}{Period}$) of the periodic signal.

Example



f(t) is periodic with period T

$$f(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos \omega 3_0 t + \cdots$$
$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin \omega 3_0 t + \cdots$$

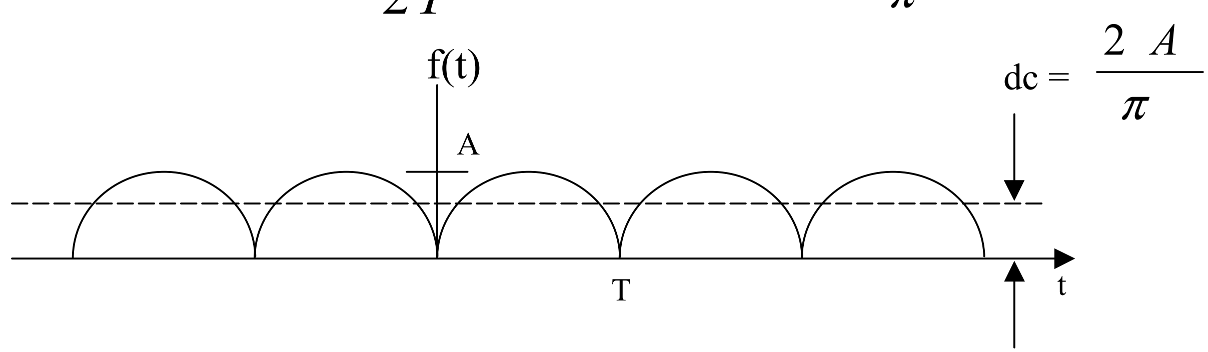
a_0, a_n, b_n, n = 1,2…., to be found. $\omega_0 = \frac{2\pi}{T}$,known

How to find a_n, b_n given f(t) ?

$$\int_0^T f(t)dt = \int_0^T a_0 dt + \underbrace{\int_0^T a_n \cos n\omega_0 t dt + \int_0^T b_n \sin n\omega_0 t dt + \cdots}_{\text{All} = 0}$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \text{dc component of f(t)}$$

Let $f(t) = A \sin \frac{2\pi}{2T} t$, then $a_0 = \frac{2A}{\pi}$



$$\int_0^T f(t)\cos\frac{2\pi}{T}t\,dt = \underbrace{\int_0^T a_0\cos\frac{2\pi}{T}t\,dt}_{=0} + \int_0^T a_1\cos\omega_0 t\cos\omega_0 t\,dt$$

$$\text{All} = 0 \quad \begin{cases} + \int_0^T a_2\cos 2\omega_0 t\cos\omega_0 t\,dt \\ + \cdots\cdots \\ + \int_0^T b_1\sin\omega_0 t\cos\omega_0 t\,dt \\ + \cdots\cdots \end{cases}$$

$$\int_0^T A\sin\frac{\pi}{T}t\cos\frac{2\pi}{T}t\,dt = a_1\int_0^T \frac{1+\cos 2\omega_0 t}{2}\,dt$$

$$a_1 = -\frac{4A\pi}{3}$$

Similarly can find all a_n, b_n, using the Orthogonal Principle.

$$\int_0^{\frac{\omega_0}{2\pi}=T} \sin n\omega_0 t \sin m\omega_0 t dt = 0 \qquad \text{for } n \neq m$$

i.e., if we multiply $\sin\omega_0 t$ and $\sin 2\omega_0 t$ and integrate over a period, the result is zero. Similarly

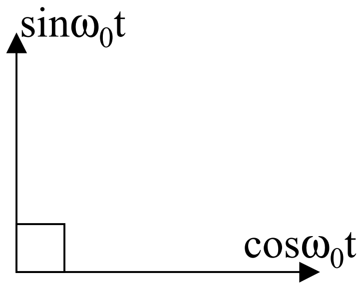
$$\int_0^T \cos n\omega_0 t \cos m\omega_0 t dt = 0 \qquad \text{for } n \neq m$$

and

$$\int_0^T \sin n\omega_0 t \cos m\omega_0 t dt = 0 \qquad \text{for any } n, m$$

if $\int_0^T f_1(t) f_2(t) dt = 0$,then

$f_1(t)$ and $f_2(t)$ are orthogonal (independent, uncorrelated) over the interval 0 to T



Orthogonal or ⊥, no common components.

Another way to see Fourier series analysis is that we can decompose a periodic f(t) into a sum of orthogonal functions, in this case, they are sines and cosines.

The reverse is also true. We can synthesize a periodic function through summation of its orthogonal components.

$$f(t) = a_0 + \sum_n a_n \cos n\omega_0 t + \sum_n b_n \sin n\omega_0 t$$

Decomposition : find a_n, b_n from f(t)

Synthesis or reconstruction : produce f(t) from $a_n \cos n\omega_0 t$ and $b_n \sin n\omega_0 t$

From $e^{j\omega t} = \cos \omega t + j \sin \omega t$

can write the Fourier series in the exponential form

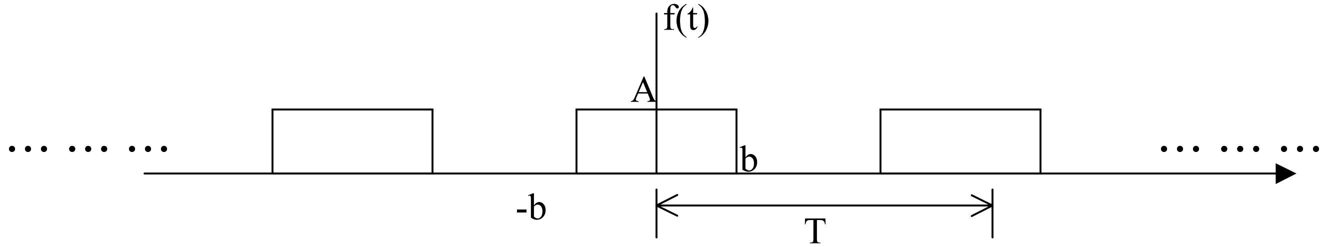
$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad \text{(synthesis)}$$

Note that since f(t) is real, F_n can be complex and we must have $F_{-n}^* = F_n$

$$F_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t}\,dt \qquad \text{(decomposition)}$$

I.2 Review Questions

- Why is Fourier series useful?
- Draw 3 periodic functions and determine their periods.
- Why do we need both sines and cosines in the Fourier series?
- Find the Fourier series representation of the function

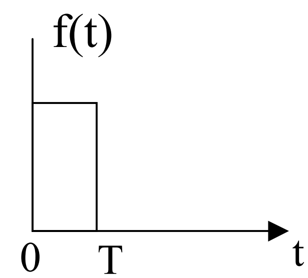


- Can you produce $e^{j2\pi\times 50t}$ in the lab?

  Can you produce –50Hz ?

I.3 Fourier Transform

Often signals are non-periodic, e.g., a pulse representing a "1". We wish to know the frequency compositions (spectrum) of this pulse.



Cannot use Fourier series.

Need Fourier transform. It decomposes a non-periodic f(t) into a <u>continuous</u> sum of sinusoids. In Fourier series, the sum is discrete and $= \sum F_n e^{jn\omega_0 t}$

Now require $\int F(\omega) e^{j\omega t} d\omega$ so that this sum will sum to zero for t > T for the pulse. This is the reason to use continuous basis functions $e^{j\omega t}$.

with ω= continuous variable, the Exponential Fourier Series representation on P. 17 becomes

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad \text{Inverse Fourier Transform}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad \text{Fourier Transform}$$

Often we need to take FT of discrete time signals, For n points of f(n), the Discrete Fourier Transform (DFT) is

$$F(\omega) = \sum_{n=0}^{N-1} f(n)e^{-j\omega n}$$

ω is still a continuous variable

$$f(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega n}d\omega$$

The Fast Fourier Transform (FFT) discretizes ω into $\omega_k = \frac{2\pi k}{N}$ is a fast way to compute the DFT. Now

$$F(k) = \sum_{n=0}^{N-1} f(n)e^{-j\frac{2\pi kn}{N}}$$

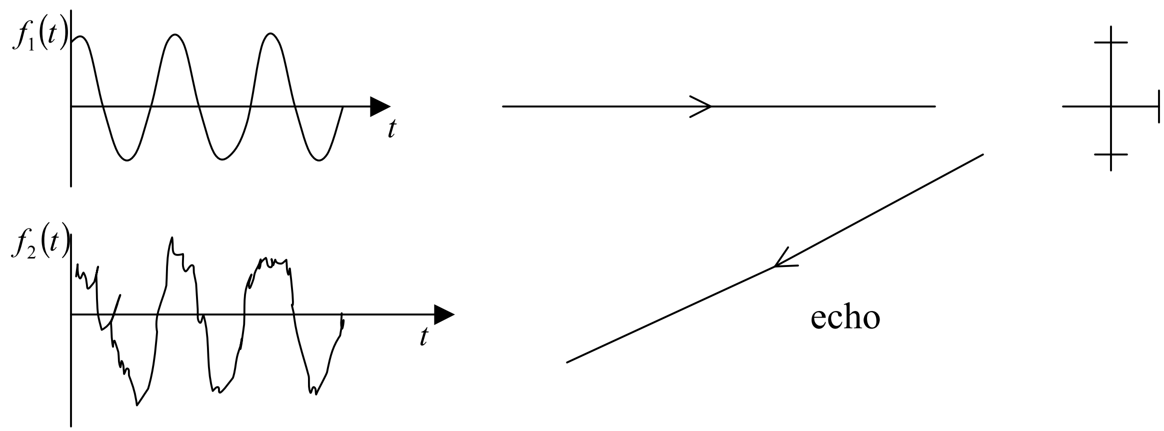
$$f(n) = \frac{1}{N}\sum_{n=0}^{N-1} F(k)e^{j\frac{2\pi kn}{N}}$$

I.3 Review Questions

- What is the difference between the Fourier series and the FT?
- Why is frequency domain analysis important?
- What are basis functions?
- What is the Orthogonal Principle?
- Do question (4) on P.4.

I.4 Correlation and Power Spectrum

Given $f_1(t)$ and $f_2(t)$, want to check if there is any relationship (similarity) between them. This has application in signal detection. In radar, the transmitter sends a sinusoidal pulse and the receiver checks for echos.



The echo $f_2(t)$ contains the pulse with noise. If there is a strong correlation between $f_1(t)$ and $f_2(t)$, the radar detects a plane.

How to check similarity ?

$$d(t) = \frac{1}{2T}\int_{-T}^{T}[f_1(t) - f_2(t)]^2\,dt$$

d(t) small if large correlation

d(t) large if weak correlation

$$2Td(t) = \int f_1^{\,2}(t)dt + \int f_2^{\,2}(t)dt - \underbrace{2\int f_1(t)f_2(t)dt}_{\text{d(t) small if this is large}}$$

In radar, the echo returns with a time delay of D.

Thus $f_2(t) = f_1(t - D)$ .Thus to check similarity, we compute

$$E[f_1(t)f_2(t+\tau)] = R_{12}(\tau) = \frac{1}{2T}\int_{-T}^{T} f_1(t)f_2(t+\tau)dt$$

$$= \frac{1}{2T}\int_{-T}^{T} f_1(t)f_1(t - D + \tau)dt$$

This is the correlation of $f_1(t)$ and $f_2(t+\tau)$, at a shift of τ.

We vary τ and compute $R_{12}(\tau)$. At τ=D, $R_{12}(\tau)$ is maximum.

Autocorrelation $R_{11}(\tau)=\frac{1}{2T}\int_{-T}^{T} f_1(t) f_1(t+\tau)dt$

Crosscorrelation $R_{12}(\tau)=\frac{1}{2T}\int_{-T}^{T} f_1(t) f_2(t+\tau)dt$

Autocorrelation checks similarity between f(t) and its time shifted version.

Useful for checking periodicity in a signal.

$$f(t)=A\cos\omega t$$

$$R(\tau)=\frac{1}{2T}\int_{-T}^{T} f(t)f(t+\tau)dt \qquad \text{,T = a period of f(t)} = \frac{2\pi}{\omega}$$

$$=\frac{A^2}{2T}\int_{-T}^{T}\cos\omega t\cos\omega(t+\tau)dt$$

$$=\frac{A^2}{2T}\int_{-T}^{T}(\cos\omega t\cos\omega t\cos\omega\tau-\cos\omega t\sin\omega t\sin\omega\tau)dt$$

$$=\frac{A^2}{2}\cos\omega\tau$$

At $\tau=0,\pm T,\pm 2T,\ldots$ $R(\tau)=\frac{A^2}{2}=$ Power if f(t) is voltage across a 1Ω resistor

$\frac{A}{\sqrt{2}}=$ RMS voltage

Power Spectrum

Take FT of the autocorrelation function

$$G_{ff}(\omega) = \int_{-\infty}^{\infty} R_{ff}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{ff}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{ff}(\omega) e^{j\omega\tau} d\omega$$

$$\underset{\text{Watts}}{R_{ff}(0)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{G_{ff}(\omega)}_{\frac{Watts}{rads/\sec}} \underbrace{d\omega}_{\text{Rad/sec}}$$

$G_{ff}(\omega)$ is power spectral density. Shows how the power of f(t) is distributed as a function of frequency. $|F(\omega)|^2$ gives also the spectrum. Hence

$$G_{ff}(\omega) = |F(\omega)|^2 = \int_{-\infty}^{\infty} R_{ff}(\tau) e^{-j\omega\tau} d\tau$$

I.4 Review Questions

- Provide two applications of the correlation function.
- What is the difference between correlation and convolution?
- If x(t) is a white noise signal, what is its autocorrelation? What is its spectrum ? What physical meanings can you derive from these quantities?
- A linear system has input x(t), impulse response h(t), and output y(t). Show that

$$Y(\omega) = H(\omega)X(\omega)$$
$$G_{yy}(\omega) = |H(\omega)|^2 G_{xx}(\omega)$$
$$R_{xy}(\tau) = R_{xx}(\tau) * h(-\tau)$$
$$R_{yy}(\tau) = R_{xy}(\tau) * h(\tau)$$
$$R_{yy}(\tau) = R_{xx}(\tau) * \rho(\tau)$$

where $\rho(\tau) =$ Inverse FT of $|H(\omega)|^2$

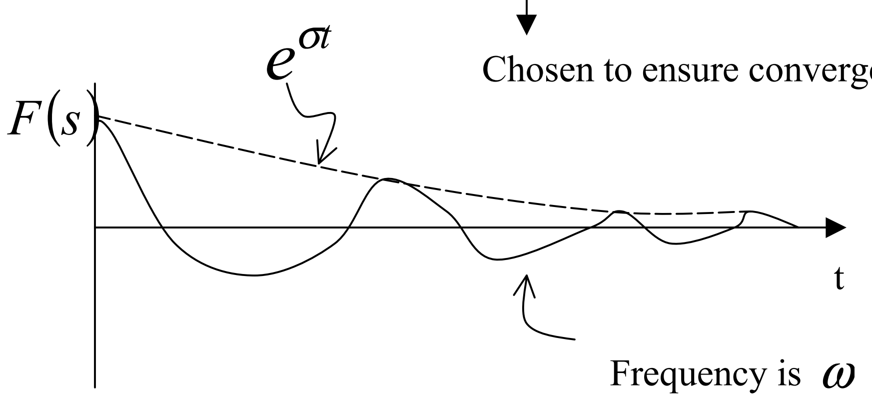
I. 5 Laplace Transform

The LT uses damped exponentials e^{st} as basis functions to model, or represent, time functions

LT: $$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$, finds the correlation between f(t) and e^{-st}

ILT: $$f(t) = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} F(s) e^{st} ds$$,reproduces f(t) as a sum of damped exponentials of amplitude F(s), damping = σ and frequency = ω

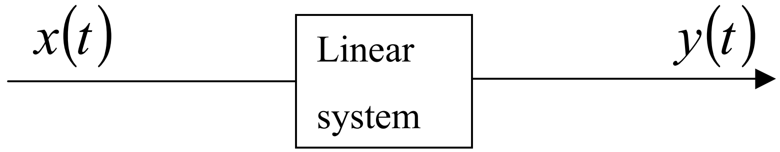
Chosen to ensure convergence



$$s = \sigma + j\omega$$

σ is negative

Linear system analysis by LT



$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = x(t) + b_1 \frac{dx(t)}{dt}$$

Consider an input $x_1(t) = X(s_1)e^{s_1 t}$,giving output $y_1(t) = Y(s_1)e^{s_1 t}$

$$s_1{}^2 Y(s_1)e^{s_1 t} + a_1 s_1 Y(s_1)e^{s_1 t} + a_0 Y(s_1)e^{s_1 t} = X(s_1)e^{s_1 t} + b_1 s_1 X(s_1)e^{s_1 t}$$

$$\frac{Y(s_1)}{X(s_1)} = \frac{1 + b_1 s_1}{s_1{}^2 + a_1 s_1 + a_0}$$

Express any x(t) as a sum of damped exponentials

$$x(t) = \frac{1}{2\pi j}\int X(s)e^{st}\,ds$$

Then by superposition, y(t) is also a sum of the individual outputs

$$y(t) = \frac{1}{2\pi j} \int Y(s) e^{st} ds = \text{Inverse LT}$$

and $$\frac{Y(s)}{X(s)} = \frac{1 + b_1 s}{s^2 + a_1 s + a_0} = \text{system transfer function}$$

Much easier to solve differential equation by LT.

Suppose in $s = \sigma + j\omega$ we let $\sigma = 0$. Then $s = j\omega$ and

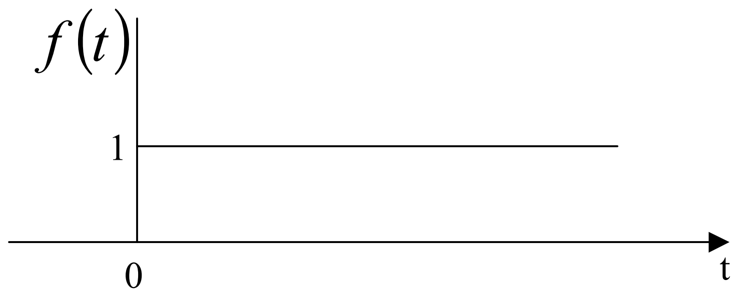
$$\frac{Y(j\omega)}{X(j\omega)} = \frac{1 + b_1 (j\omega)}{(j\omega)^2 + a_1 (j\omega) + a_0} = H(\omega)$$

which gives the frequency response of the system.

I.5 Review Questions

- Do Questions (1) and (2) on P.3
- What is the difference between LT and FT ?
- Find the LT of the functions

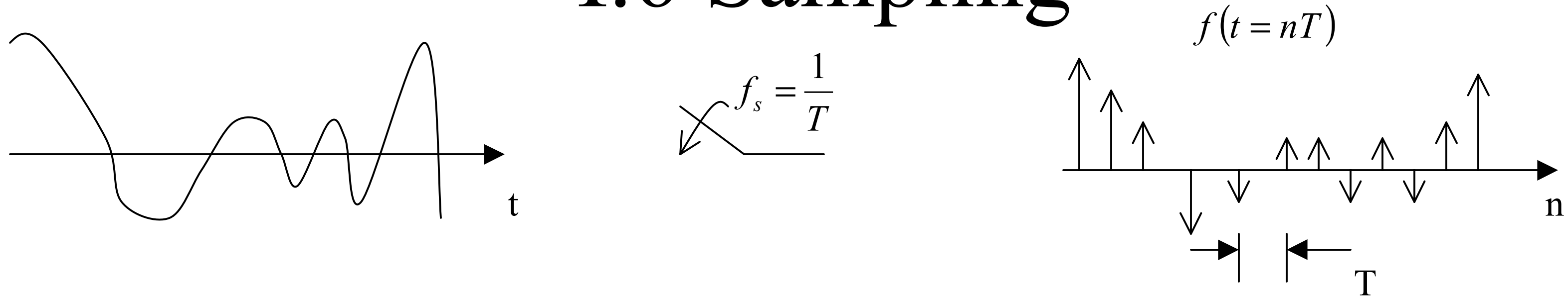
a)



a step function

b) $f(t) = \cos \omega_0 t$

I.6 Sampling

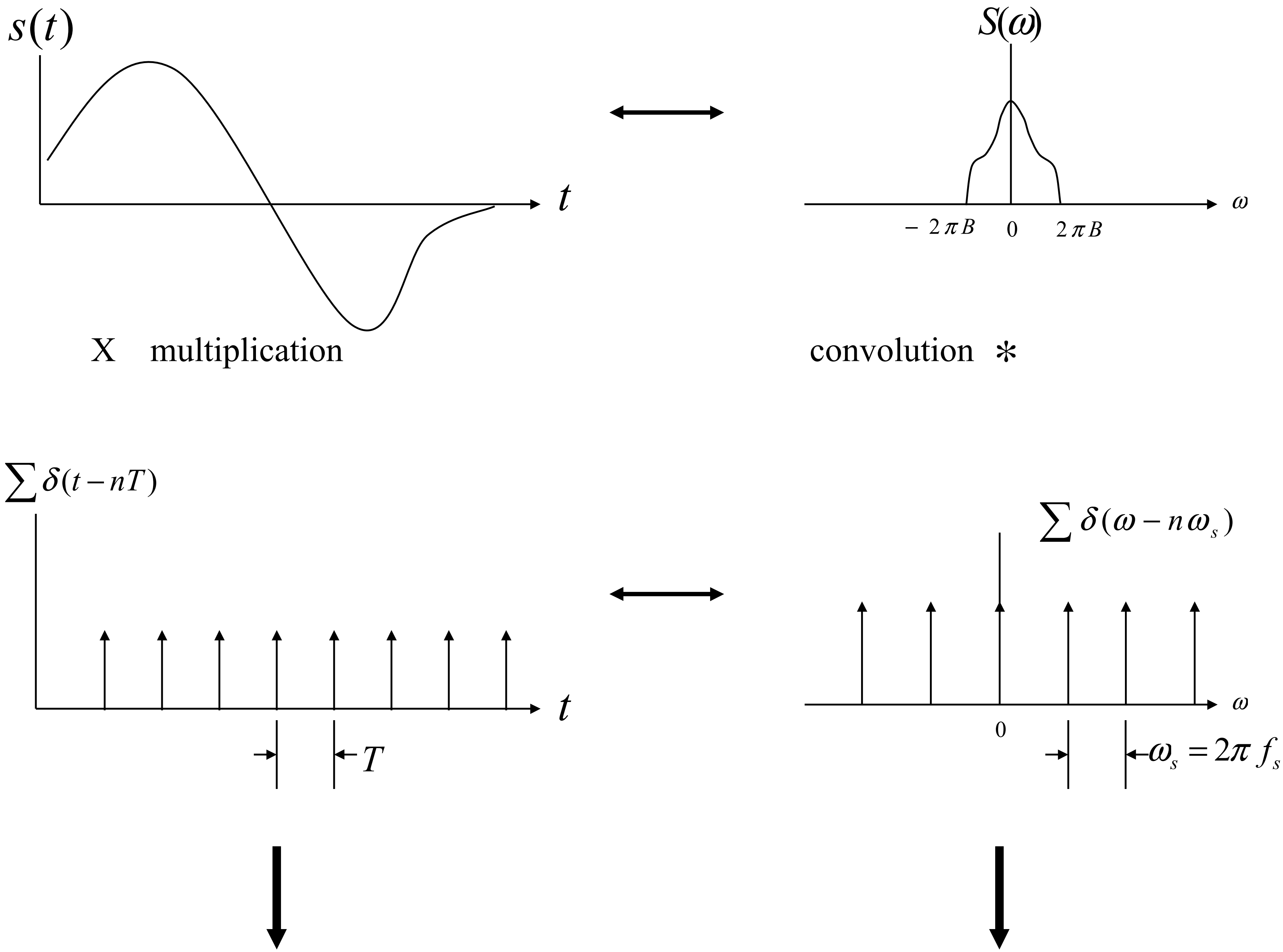


f_s = sampling frequency

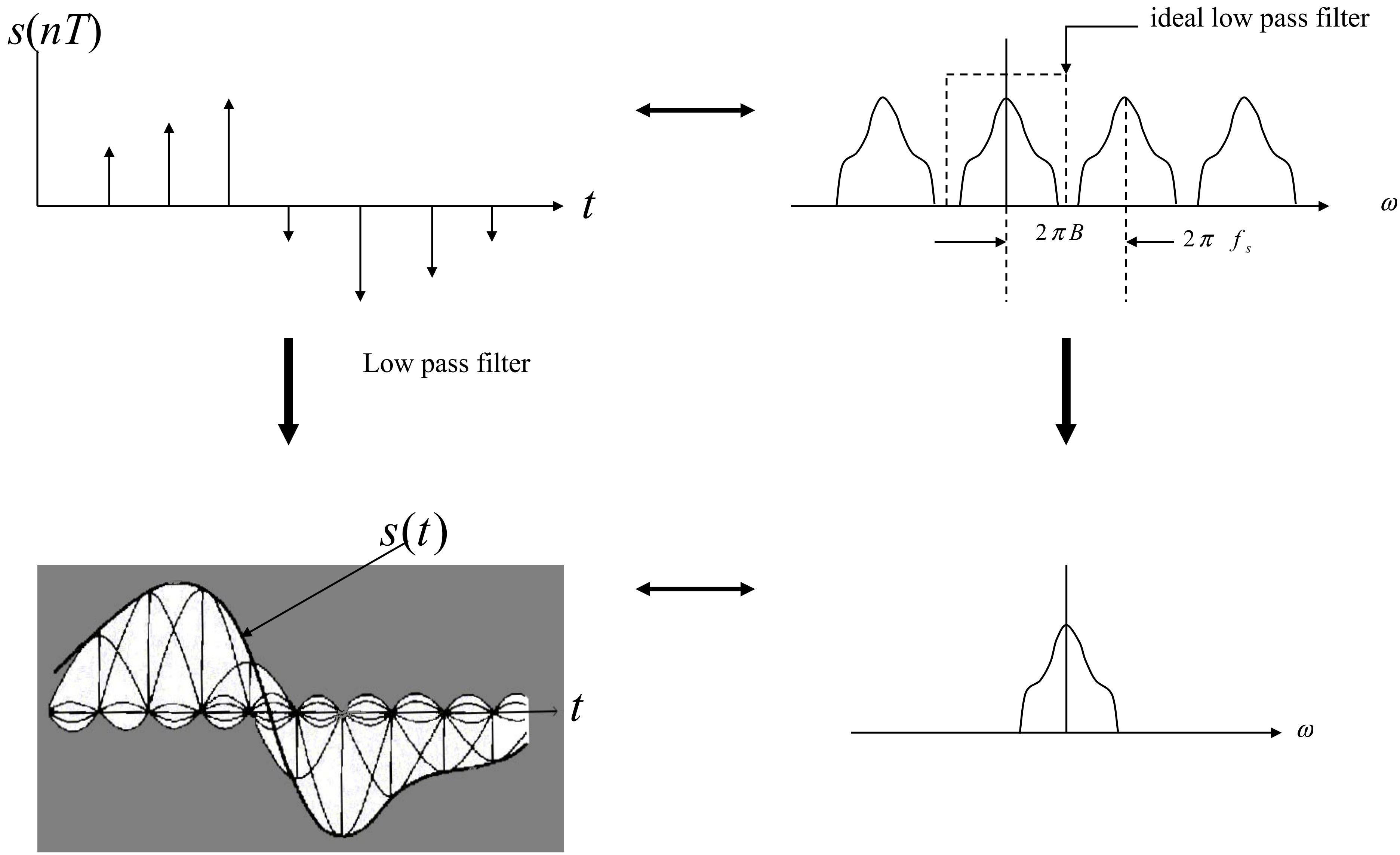
T = sampling period

Sampling basics

- There is redundancy of information in a bandlimited analogue signal.
- Samples of f(t), $f(nT)$, can contain the same amount of information as f(t). Thus can reproduce f(t) from $f(nT)$.
- Must take samples at sampling frequency for $f_s > f_u$, where f_u is the bandwidth of the signal.



$s(t)$
t
$S(\omega)$
ω
$-2\pi B$
0
$2\pi B$
X multiplication
convolution *
$\sum \delta(t-nT)$
t
T
$\sum \delta(\omega - n\omega_s)$
ω
0
$\omega_s = 2\pi f_s$


$s(nT)$
t
ideal low pass filter
ω
$2\pi B$
$2\pi f_s$
Low pass filter
$s(t)$
t
ω

I.6 Review Questions

- Why should we do sampling?
- We always low pass filter a signal before sampling. Why?
- How do we reconstruct f(t) from its samples $f(nT)$?
- What happens if we sample at $f_s < f_u$?

I.7 Analogue Modulation

Why modulation?

- Many transmitters share the same communication channel. Modulation permits receivers to select a particular transmission.
- Need to transmit signals at frequencies > 100 kHz so that antenna lengths are reasonable.

Want antenna length > $\frac{1}{10}$ wavelength

Transmit voice at 4kHz, length > $\frac{3 \times 10^{8}}{10 \times 4 \times 10^{3}} \approx 10^{4}\ m$, too long

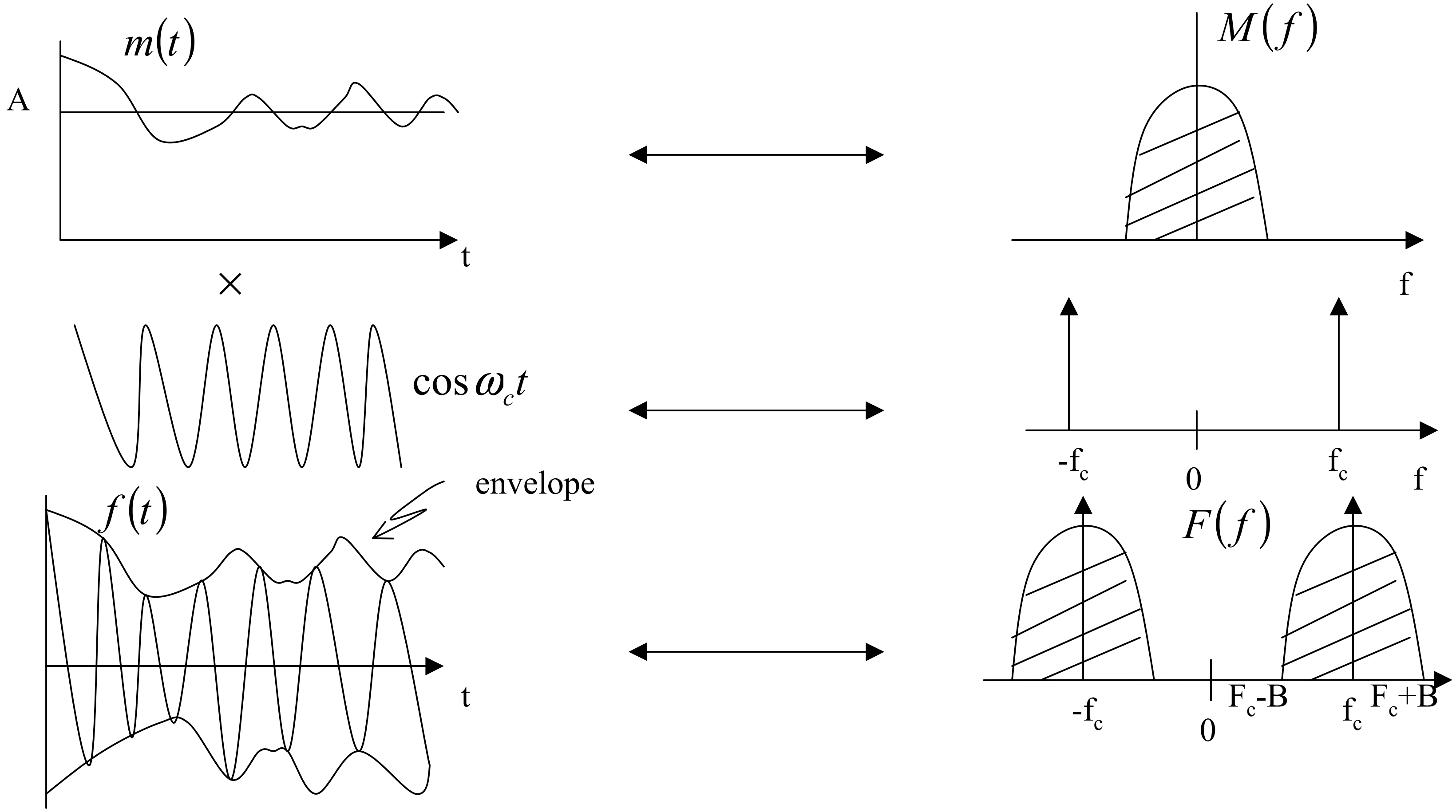
Translate to 100kHz, length > $\frac{3 \times 10^{8}}{10 \times 100 \times 10^{3}} \approx 300\ m$, O.K.

Amplitude Modulation (AM)

Message (voice or music) is m(t)

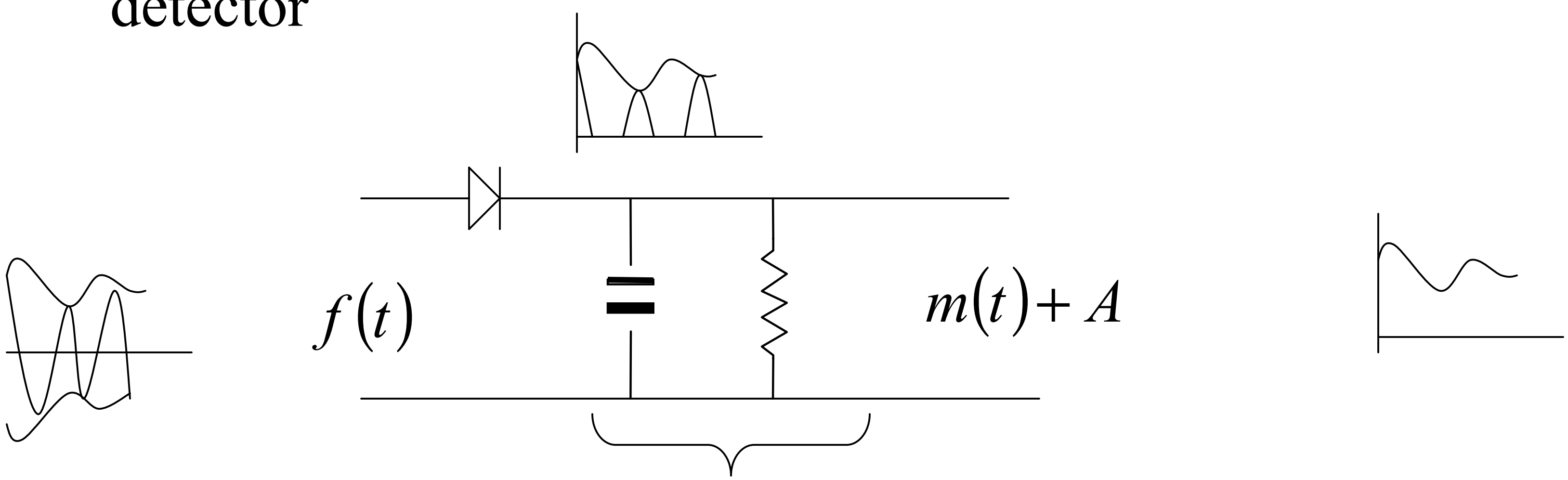
carrier is $\cos\omega_c t$

The modulated signal for transmission is $f(t) = (m(t) + A)\cos\omega_c t$



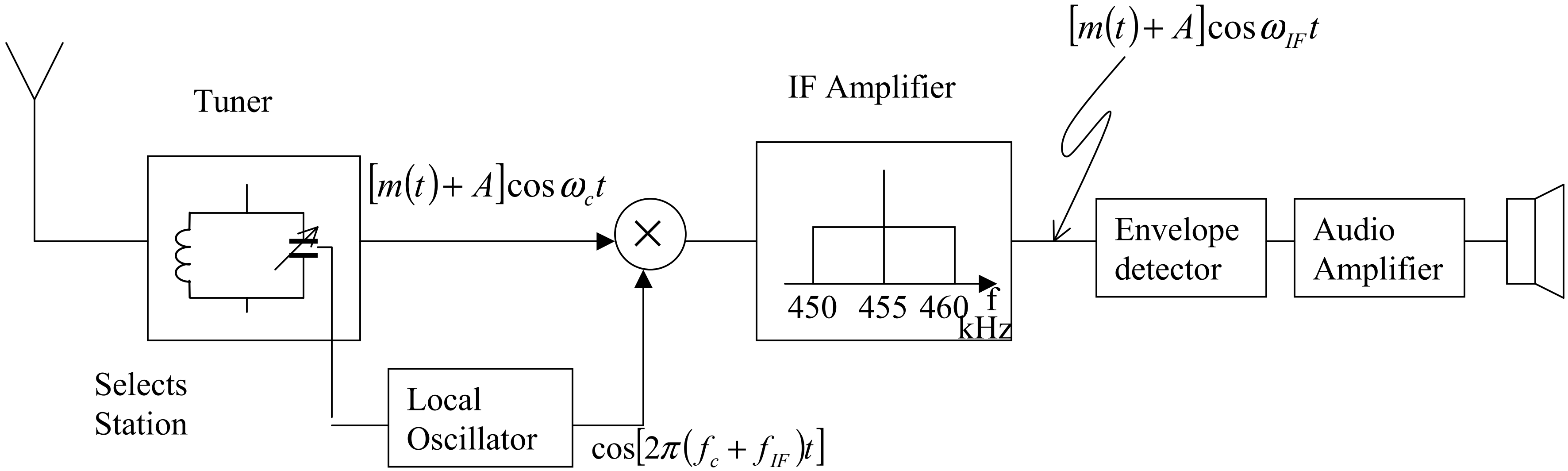
In AM, m(t) changes the amplitude (envelope) of the carrier according to its own variations.

A demodulator that recovers m(t) from f(t) is the envelope detector



Low pass filter passes all M(f) inside the bandwidth B but rejects the carrier f_c

The Superhetrodyne Receiver



f_{IF} = intermediate frequency = 455kHz

- Select station by varying capacitance in tuner, which is a bandpass filter of centre frequency $=\frac{1}{\sqrt{LC}}$. Tuning knob also changes frequency of local oscillator to output frequency $= f_c + f_{IF}$
- Multiplication of two frequencies gives sum and difference of the frequencies. The IF amplifier selects the difference.

Frequency Modulation (FM) is when m(t) changes the carrier frequency according to its own variations.

$$f(t) = \cos\underbrace{\left[\omega_c t + k\int_0^t m(\lambda)d\lambda\right]}_{\theta(t)}$$

k = modulation constant

instantaneous frequency $\omega_i(t) = \dfrac{d\theta(t)}{dt} = \omega_c + km(t)$

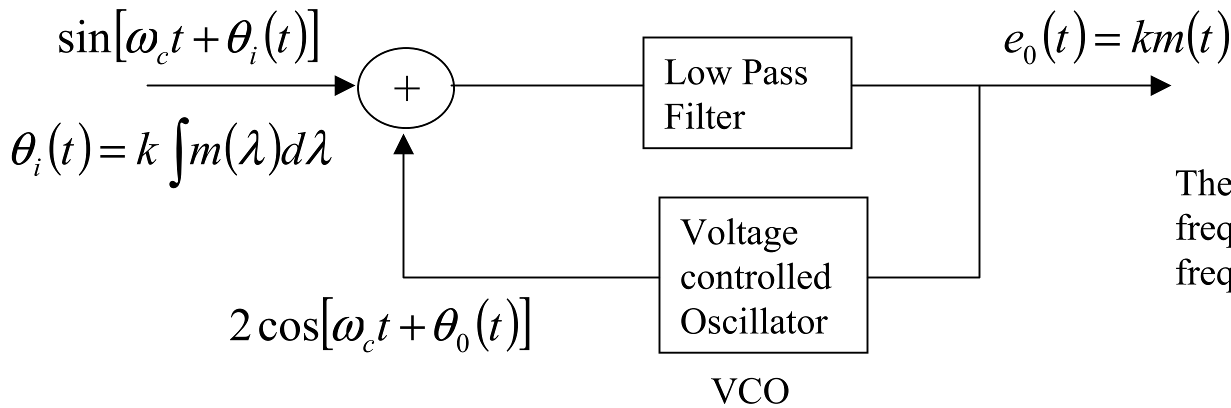
Thus ω_i(t) changes according to changes in km(t)

amount of change = $|km(t)|$

rate of change = rate of change of m(t)

Demodulation of FM is by Phase Locked Loop

It operates similar to a feedback control system



The VCO integrate $e_0(t)$ to produce a frequency that tracks the input frequency.

$$2\sin[\omega_c t + \theta_i(t)]\cos[\omega_c t + \theta_0(t)] = \underbrace{\sin[2\omega_c t + \theta_i(t) + \theta_0(t)]}_{\text{LPF rejects}} + \underbrace{\sin[\theta_i(t) - \theta_0(t)]}_{\text{Phase error } \approx \theta_i(t) - \theta_0(t)}$$

$e_0(t)$ drives VCO to produce a zero frequency error.

Hence frequency error = $\dfrac{d(Phase\ error)}{dt} = \dot{\theta}_i(t) - \dot{\theta}_0(t) = 0 \Rightarrow \dot{\theta}_0(t) = km(t)$

but $\int e_0(t)dt = \theta_0(t)$,hence $e_0(t) = km(t)$

I.7 Review Questions

- What is the difference between AM and FM ?
- In AM, the spectrum from f_c to $f_c + B$ is the same as from $f_c - B$ to f_c . It is therefore necessary to use the bandwidth 2B. For TV when the BW is in MHz, it is important to save BW. This leads to Single Side Band systems. List the differences and compare AM to SSB.
- To save power, sometimes "Suppressed carrier" is used, i.e. the carrier is not transmitted. This requires non-coherent detection. Explain the difference between coherent and non-coherent detection.